17. The Monodromy Group and the Reducibility Conditions of the One Dimensional Section of Appell's Hypergeometric Equation for $F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y)$

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(Communicated by Kôsaku YOSIDA, M. J. A., March 13, 1978)

In this paper we announce certain results on the monodromy group and the reducibility conditions of the one dimensional section of the system of partial differential equations for Appell's hypergeometric function F_3:

$$\sum_{k=0}^{4} A_{4-k}(t)\frac{d^k x}{dt^k}=0, \tag{1}$$

where

$$\begin{aligned}
&A_0=t^2(t-1)(t-c),\\
&A_1=t[(2\gamma-\alpha'-\beta'+3)(t-1)(t-c)+(\alpha+\beta-\gamma+3)t(t-c)\\
&\qquad+(\alpha+\beta+\alpha'+\beta'+2-\gamma)t(t-1)],\\
&A_2=[(\alpha+\beta+3)t-(\gamma+1)][(\alpha+\beta+3)t+(\alpha'+\beta'-\gamma-1)c]\\
&\qquad+t(t-c)(2\alpha\beta+3\alpha+3\beta+5)+(c-1)(\alpha+1)(\beta+1)t+\alpha'\beta'c,\\
&A_3=(\alpha+1)(\beta+1)[(2\alpha+2\beta+4)t+(\alpha'+\beta'-\gamma-1)c-\gamma],\\
&A_4=\alpha\beta(\alpha+1)(\beta+1),
\end{aligned}$$

and $c\neq 0, 1$. This equation was first studied by Goursat [2], more precisely, see [1], p. 72 and p. 87. We can rewrite it in the following form:

$$(tI-B)\frac{dx}{dt}=Ax, \tag{2}$$

where I is the unit matrix of degree 4, $A=(a_{ij})$ and the diagonal matrix $B=\mathrm{diag}\,(0, 0, 1, c)$ are 4 by 4 constant matrices:

$$a_{11}=\alpha'-\gamma+1,\ a_{22}=\beta'-\gamma,\ a_{33}=\gamma-\alpha-\beta-1,\ a_{44}=\gamma-\alpha-\beta-\alpha'-\beta',$$

$$a_{12}=a_{21}=0,\ a_{13}=a_{14}=-a_{32}=a_{42}=1,$$

$$a_{23}=\frac{\alpha'}{\alpha'-\beta'+1}(\alpha+\beta'-\gamma)(\beta+\beta'-\gamma),\ a_{24}=\frac{\beta'-1}{\alpha'-\beta'+1}(\alpha+\beta'-\gamma)(\beta+\beta'-\gamma),$$

$$a_{31}=\frac{\beta'-1}{\alpha'-\beta'+1}(\alpha+\alpha'-\gamma+1)(\beta+\alpha'-\gamma+1),\ a_{34}=\beta'-1,$$

$$a_{41}=-\frac{\alpha'}{\alpha'-\beta'+1}(\alpha+\alpha'-\gamma+1)(\beta+\alpha'-\gamma+1),\ a_{43}=-\alpha',$$

and $x=(x_1, x_2, x_3, x_4)$ is a 4-vector. In fact, eliminating the components x_2, x_3 and x_4 and replacing x_1 by x, we obtain the equation (1).

From now on we only consider the system (2). It is a Fuchsian differential equation with four regular singular points $t=0, 1, c$ and ∞ the Riemann sphere S^2. The characteristic exponents are $(a_{11}, a_{22}, 0, 0)$ at $t=0$, $(0, 0, a_{33}, 0)$ at $t=1$, $(0, 0, 0, a_{44})$ at $t=c$ and $(-\alpha, -\alpha, -\beta, -\beta)$ at $t=\infty$. We note that $-\alpha$ and $-\beta$ are two eigenvalues of the matrix A with the common multiplicity 2. In this case Fuchs' relation is merely the invariance of the trace of A.

Theorem 1 (Fuchs' relation). $-2(\alpha+\beta)=\sum_{j=1}^{4} a_{jj}$.

Now we assume that *none of the quantities* a_{jj} $(j=1,2,3,4)$, $a_{jj}-a_{kk}$ $(j\neq k)$, α, β *and* $\alpha-\beta$ *is an integer. Consequently there is no logarithmic solution.*

Under these circumstances we can prove

Theorem 2. *The system* (2) *is accessary parameter free.*

Theorem 3. *The system* (2) *has only four singular solutions defined at three finite singular points* $t=0, 1$ *and* c, *corresponding to the characteristic exponents* a_{jj} *and the normalization conditions* $g_j(0)=\varepsilon_j$:

$$X_j(t)=(t-t_j)^{a_{jj}}\cdot\sum_{m=0}^{\infty} g_j(m)\cdot(t-t_j)^m \qquad (j=1,2,3,4),$$

where the j-th component and the others of the 4-*vector* ε_j *are* 1 *and* 0, *respectively, and* $t_1=t_2=0$, $t_3=1$ *and* $t_4=c$.

The formula of the following type was first investigated by Okubo [3] which is an extension of Gauss' formula in the theory of the classical hypergeometric equation. With a slight modification of Okubo's proof, we obtain

Theorem (Okubo) **4.** *In any simply connected domain contained in* $S^2-\{0, 1, c, \infty\}$, *the Wronskian of the above solutions is*

$$\det X=\frac{\prod_{j=1}^{4}(t-t_j)^{a_{jj}}\cdot\Gamma(a_{jj}+1)}{[\Gamma(1-\alpha)\cdot\Gamma(1-\beta)]^2},$$

where X *is the matrix* (X_1, X_2, X_3, X_4), *from which the linear independence of the solutions follows.*

Now we consider the monodromy group with respect to the basis X. First we fix a base point t_0 in $S^2-\{0, 1, c, \infty\}$. Let μ_j $(j=1,3,4)$ be a simple loop which start at t_0, go around t_j once in the positive direction and return to t_0. We may choose t_0 and μ_j in such a way that the composition $\mu_\infty=\mu_1\cdot\mu_3\cdot\mu_4$ is a simple loop surrounding $t=\infty$ in the negative direction, where $\mu_k\cdot\mu_j$ is the loop μ_j followed by μ_k. The loops $\{\mu_j\}$ generate the fundamental group $\pi(S^2-\{0, 1, c, \infty\}, t_0)$. If we continue analytically the basis X along μ_j, X is transformed into XM_j, where M_j is an element of $GL(4, \boldsymbol{C})$ and is called the *monodromy matrix*. As is well-known, M_j $(j=1,3,4)$ generate the monodromy group of the system (2).

Finally we state the two main theorems. Let e_i, f_1 and f_2 be $\exp(2\pi i a_{jj})$, $\exp(-2\mu i\alpha)$ and $\exp(-2\pi i\beta)$, respectively.

Theorem 5. *M_j can be determined as follows.*

$$M_1=\begin{bmatrix} e_1 & 0 & \dfrac{e_2(e_1-f_1)(e_1-f_2)}{f_1f_2(e_1-e_2)} & \dfrac{(e_1-f_1)(e_1-f_2)(e_1e_3-f_1f_2)}{e_1e_3(e_2-e_1)} \\ 0 & e_2 & \dfrac{e_1(e_2-f_1)(e_2-f_2)}{f_1f_2(e_2-e_1)} & \dfrac{(e_2-f_1)(e_2-f_2)(e_2e_3-f_1f_2)}{e_2e_3(e_1-e_2)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_3=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \dfrac{e_2e_3-f_1f_2}{e_2} & \dfrac{e_1e_3-f_1f_2}{e_1} & e_3 & \dfrac{(e_1e_3-f_1f_2)(f_1f_2-e_2e_3)}{e_1e_2e_3} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_4=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & e_4 \end{bmatrix}.$$

Theorem 6. *The following conditions are equivalent:*

(a) *The system* (2) *is reducible.*

(b) *The monodromy group of the system* (2) *is reducible.*

(c) *One of the non-trivial components stated in Theorem* 5 *is zero.*

(d) *One of the following six quantities:*

$$\alpha',\ \beta',\ \alpha+\alpha'-\gamma,\ \beta+\alpha'-\gamma,\ \alpha+\beta'-\gamma,\ \beta+\beta'-\gamma,$$

is an integer.

The detail will be given in [4].

References

[1] P. Appell et J. Kampé de Fériet: Fonctions Hypergéométriques et Hypersphériques. Polynomes d'Hermite, Gauthier-Villars, Paris (1926).

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[4] T. Sasai: On a monodromy group and irreducibility conditions of a fourth order Fuchsian differential system of Okubo type (to appear in J. Reine Angew. Math.).