6. A Note on Normal Ideals of Asano Orders

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Let R be a ring with an identity and let $\mathfrak{O}$ be an Asano order of R. It is well-known that the set of all two-sided $\mathfrak{O}$-ideals forms a group under multiplication and every integral two-sided $\mathfrak{O}$-ideal can be uniquely written as a product of a finite number of prime ideals of $\mathfrak{O}$ ([1]).

In the present paper we show that, if two Asano orders $\mathfrak{O}$ and $\mathfrak{O}'$ of R are equivalent to each other, then the set of all left $\mathfrak{O}$- and right $\mathfrak{O}'$-ideals ($\mathfrak{O}-\mathfrak{O}'$-ideals) forms a group under multiplication which is isomorphic to the group of all two-sided $\mathfrak{O}$-ideals. Then the concept of prime ideals is naturally generalized for $\mathfrak{O}-\mathfrak{O}'$-ideals. By using the definition we obtain prime-factorization of $\mathfrak{O}-\mathfrak{O}'$-ideals.

Our results can be generalized in the case of semigroup treated in [2].

Let $\mathfrak{O}$ be an order of a ring R with an identity. A subset A of R is called a left $\mathfrak{O}$-ideal if (i) A is a left $\mathfrak{O}$-module, (ii) A contains a regular element and (iii) $Aa \subseteq \mathfrak{O}$ for some regular element a of R. Right $\mathfrak{O}'$-ideals are defined in a similar fashion, where $\mathfrak{O}'$ denotes an order of R. If A is a left $\mathfrak{O}$-ideal and a right $\mathfrak{O}'$-ideal, then A is called an $\mathfrak{O}-\mathfrak{O}'$-ideal. Let $\{\mathfrak{O}^i, \mathfrak{O}^k, \cdots\}$ be a system of the orders which are equivalent to a fixed order $\mathfrak{O}$ of R.

Throughout this paper we shall assume that:

(1) $\mathfrak{O}^i, \mathfrak{O}^k, \cdots$ are maximal orders of R.

(2) A fixed order $\mathfrak{O}^i$ is regular (bounded).

(3) Ascending chain condition holds for integral two-sided $\mathfrak{O}^i$-ideals for a fixed order $\mathfrak{O}^i$.

(4) Each prime ideal of $\mathfrak{O}^i$ is maximal for a fixed order $\mathfrak{O}^i$.

Let $\mathfrak{N}_{ik}$ be the set of all $\mathfrak{O}^i-\mathfrak{O}^k$-ideals. For convenience $A = A^{ik}$ will denote that $A \in \mathfrak{N}_{ik}$. The product AB of an $\mathfrak{O}^i-\mathfrak{O}^k$-ideal A and an $\mathfrak{O}^j-\mathfrak{O}^l$-ideal B is called proper if $k=j$. Then the set of all ideals defined on the system $\{\mathfrak{O}^i, \mathfrak{O}^k, \cdots\}$ forms the Brandt's groupoid under proper multiplication.

Lemma 1. *Let A and B be an $\mathfrak{O}^i-\mathfrak{O}^k$-ideal and an $\mathfrak{O}^j-\mathfrak{O}^l$-ideal respectively. Then the product AB, which is not necessarily proper, is an $\mathfrak{O}^i-\mathfrak{O}^l$-ideal.*

Proof. It is clear that $\mathfrak{O}^i AB \mathfrak{O}^l \subseteq AB$ and AB contains a regular element. Let a and b be regular elements of R such that $Aa\subseteq\mathfrak{O}^i$ and $Bb\subseteq\mathfrak{O}^j$. By the regularity of the orders $\mathfrak{O}^k$ and $\mathfrak{O}^j$, there exists a regular element c of R such that $\mathfrak{O}^j c\subseteq\mathfrak{O}^k$. Then we have $ABbca \subseteq A\mathfrak{O}^j ca\subseteq A\mathfrak{O}^k a=Aa\subseteq\mathfrak{O}^i$. Thus AB is a left $\mathfrak{O}^i$-ideal. Similarly we obtain that AB is a right $\mathfrak{O}^l$-ideal.

Theorem. 1. *The set $\mathfrak{R}_{ik}$ of all $\mathfrak{O}^i-\mathfrak{O}^k$-ideals forms an abelian group under multiplication for any fixed indices i and k. Moreover $\mathfrak{R}_{ik}$ is isomorphic to the group $\mathfrak{R}_{ii}$ of all two-sided $\mathfrak{O}^i$-ideals.*

Proof. By Lemma 1, it is clear that $\mathfrak{R}_{ik}$ forms a semigroup under multiplication. Let A be any $\mathfrak{O}^i-\mathfrak{O}^k$-ideal. Evidently $(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$ is an $\mathfrak{O}^i-\mathfrak{O}^k$-ideal. Then we have that $A(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=A\mathfrak{O}^k\mathfrak{O}^i(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=A\mathfrak{O}^k=A$ and $(\mathfrak{O}^k\mathfrak{O}^i)^{-1}A=(\mathfrak{O}^k\mathfrak{O}^i)^{-1}\mathfrak{O}^k\mathfrak{O}^i A=\mathfrak{O}^i A=A$. Thus $(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$ is an identity of $\mathfrak{R}_{ik}$. Again by Lemma 1, $(A\mathfrak{O}^i)^{-1}(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$ is an $\mathfrak{O}^i-\mathfrak{O}^k$-ideal. And we have that $A(A\mathfrak{O}^i)^{-1}(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=A\mathfrak{O}^i(A\mathfrak{O}^i)^{-1}(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=\mathfrak{O}^i(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$ and $(A\mathfrak{O}^i)^{-1}(\mathfrak{O}^k\mathfrak{O}^i)^{-1}A=(A\mathfrak{O}^i)^{-1}A=(A\mathfrak{O}^i)^{-1}A\mathfrak{O}^k=(A\mathfrak{O}^i)^{-1}A\mathfrak{O}^k\mathfrak{O}^i(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=(A\mathfrak{O}^i)^{-1}A\mathfrak{O}^i(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=\mathfrak{O}^i(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$. Thus A has an inverse in $\mathfrak{R}_{ik}$. For the second part, we shall define the mapping $f:\mathfrak{R}_{ii}\to\mathfrak{R}_{ik}$ given by $f(C^{ii})=C(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$. If $C^{ii}(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=D^{ii}(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$, then we have $C=C\mathfrak{O}^i=C(\mathfrak{O}^k\mathfrak{O}^i)^{-1}\mathfrak{O}^k\mathfrak{O}^i=C(\mathfrak{O}^k\mathfrak{O}^i)^{-1}\mathfrak{O}^i=D(\mathfrak{O}^k\mathfrak{O}^i)^{-1}\mathfrak{O}^i=D$; hence f is injective. Let A be any $\mathfrak{O}^i-\mathfrak{O}^k$-ideal. Then we have $A^{ik}=A(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=A\mathfrak{O}^i(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$. Since $A\mathfrak{O}^i\in\mathfrak{R}_{ii}$, f is surjective. And we have $(C^{ii}(\mathfrak{O}^k\mathfrak{O}^i)^{-1})(D^{ii}(\mathfrak{O}^k\mathfrak{O}^i)^{-1})=C((\mathfrak{O}^k\mathfrak{O}^i)^{-1}\cdot(D(\mathfrak{O}^k\mathfrak{O}^i)^{-1}))=C(D(\mathfrak{O}^k\mathfrak{O}^i)^{-1})=(CD)(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$. Thus $\mathfrak{R}_{ii}$ is isomorphic to $\mathfrak{R}_{ik}$ as a group. Since the group $\mathfrak{R}_{ii}$ is abelian, so is $\mathfrak{R}_{ik}$. The proof is complete.

Now we define prime ideals in $\mathfrak{R}_{ik}$ as follows:

Definition. An $\mathfrak{O}^i-\mathfrak{O}^k$-ideal Q is called *prime* if (1) $Q\subset(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$ and (2) $A^{ik}B^{ik}\subseteq Q$ implies $A\subseteq Q$ or $B\subseteq Q$, where $A\subseteq(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$ and $B\subseteq(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$.

Lemma 2. *If P^{ii} is prime ideal in $\mathfrak{R}_{ii}$, then $P(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$ is a prime ideal in $\mathfrak{R}_{ik}$. Conversely, if Q^{ik} is a prime ideal in $\mathfrak{R}_{ik}$, then $Q\mathfrak{O}^i$ is a prime ideal in $\mathfrak{R}_{ii}$.*

Proof. Suppose that $A^{ik}B^{ik}\subseteq P(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$, where $A\subseteq(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$ and $B\subseteq(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$. Then we have $A\mathfrak{O}^i B\mathfrak{O}^i=AB\mathfrak{O}^i\subseteq P(\mathfrak{O}^k\mathfrak{O}^i)^{-1}\mathfrak{O}^i=P(\mathfrak{O}^k\mathfrak{O}^i)^{-1}\cdot\mathfrak{O}^k\mathfrak{O}^i=P\mathfrak{O}^i=P$. Since P is prime in $\mathfrak{R}_{ii}$, we have that $A\mathfrak{O}^i\subseteq P$ or $B\mathfrak{O}^i\subseteq P$. This implies that $A=A\mathfrak{O}^k=A\mathfrak{O}^k\mathfrak{O}^i(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=A\mathfrak{O}^i(\mathfrak{O}^k\mathfrak{O}^i)^{-1}\subseteq P(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$ or $B\subseteq P(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$. Suppose that $C^{ii}D^{ii}\subseteq Q^{ik}\mathfrak{O}^i$, where C and D are integral in $\mathfrak{R}_{ii}$. Then we have $C(\mathfrak{O}^k\mathfrak{O}^i)^{-1}D(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=CD(\mathfrak{O}^k\mathfrak{O}^i)^{-1}\subseteq Q\mathfrak{O}^i(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=Q$. Since Q is prime in $\mathfrak{R}_{ik}$, we have that $C(\mathfrak{O}^k\mathfrak{O}^i)^{-1}\subseteq Q$ or $D(\mathfrak{O}^k\mathfrak{O}^i)^{-1}\subseteq Q$. This implies $C=C(\mathfrak{O}^k\mathfrak{O}^i)^{-1}\mathfrak{O}^k\mathfrak{O}^i=C(\mathfrak{O}^k\mathfrak{O}^i)^{-1}\mathfrak{O}^i\subseteq Q\mathfrak{O}^i$ or $D\subseteq Q\mathfrak{O}^i$.

By Theorem 1 and Lemma 2, we have

Theorem 2. *Every $\mathfrak{O}^i-\mathfrak{O}^k$-ideal contained in $(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$ can be uniquely written as a product of a finite number of prime ideals in $\mathfrak{R}_{ik}$. Moreover if $A^{ik}\mathfrak{O}^i=P_1^{ii}\cdots P_n^{ii}$ is the prime-factorization of $A\mathfrak{O}^i$ in $\mathfrak{R}_{ii}$, then $A^{ik}=(P_1(\mathfrak{O}^k\mathfrak{O}^i)^{-1})\cdots(P_n(\mathfrak{O}^k\mathfrak{O}^i)^{-1})$ is the prime-factorization of A in $\mathfrak{R}_{ik}$ and, if $C^{ii}(\mathfrak{O}^k\mathfrak{O}^i)^{-1}=Q_1^{ik}\cdots Q_m^{ik}$ is the prime-factorization of $C(\mathfrak{O}^k\mathfrak{O}^i)^{-1}$ in $\mathfrak{R}_{ik}$, then $C^{ii}=(Q_1\mathfrak{O}^i)\cdots(Q_m\mathfrak{O}^i)$ is the prime-factorization of C in $\mathfrak{R}_{ii}$.*

References

[1] K. Asano: The Theory of Rings and Ideals. Tokyo (1949) (in Japanese).
[2] K. Asano and K. Murata: Arithmetical ideal theory in semigroups. Journ. Institute of Polytec. Osaka City Univ., Series A, 4 (1953).