5. An Approach to Linear Hyperbolic Evolution Equations by the Yosida Approximation Method

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§ 1. Introduction. T. Kato [1, 2] studied the Cauchy problem for linear "hyperbolic" evolution equations in a general Banach space X:

(1.1) $\quad (du/dt)+A(t)u(t)=0, \quad u(s)=x, \quad 0\leq s\leq t\leq T<\infty,$

where $-A(t)$ is the generator of a (C_0)-semigroup in X for each t. He proved the basic existence theorem [1; Theorem 4.1] by the Cauchy's method analogous to ordinary differential equations. He posed a question whether it is possible or not to prove the theorem by the Yosida approximation method. In this paper we will answer the question affirmatively under the assumptions of Kato [1; Theorem 4.1]. In § 2 we treat the "stable" case about the family $\{A(t)\}$; we study some properties of the Yosida approximation, then in § 3 we prove the existence theorem. Finally in § 4 we give some comments how our arguments are modified in the case of "quasi-stability" [2].

§ 2. Theorem. We follow Kato [1] in notation and terminology. Let X and Y be real Banach spaces with Y densely and continuously embedded in X. We assume that $-A(t)$ is the generator of a (C_0)-semigroup on X. Further assume

(i) $\{A(t)\}$ is stable; i.e., there are constants M, β such that:

$$\|(A(t_k)+\lambda)^{-1}\cdots(A(t_1)+\lambda)^{-1}\|\leq M\cdot(\lambda-\beta)^{-k}$$

for $\lambda>\beta$ and $0\leq t_1\leq\cdots\leq t_k\leq T$, $k=1,2,\cdots$.

(ii) Y is $A(t)$-admissible for each t; that is, the semigroup generated by $-A(t)$ leaves Y invariant and forms a (C_0)-semigroup on Y. And if $\tilde{A}(t)$ is the part of $A(t)$ in Y, then $\{\tilde{A}(t)\}$ is stable with some constants $\tilde{M}, \tilde{\beta}$ [1, p. 242].

(iii) $Y\subset D(A(t))$ for each t and $A(t)$ is norm continuous from $[0, T]$ into $B(Y, X)$.

Hereafter we assume $\beta, \tilde{\beta}>0$ for simplicity.

A family $\{U(t,s); 0\leq s\leq t\leq T\}$ is called the evolution operator for $\{A(t)\}$ if it satisfies the following conditions:

(a) $U(t,s)$ is strongly continuous (X) in s, t and, $U(t,t)=I$ and $\|U(t,s)\|\leq M\cdot\exp[\beta(t-s)]$.

(b) $U(t,r)=U(t,s)U(s,r)$, $r\leq s\leq t$.

(c) $(\partial/\partial t)^{+}U(t,s)y|_{t=s}=-A(s)y$ for $y\in Y$, $0\leq s<T$.

(d) $(\partial/\partial s)U(t,s)y=U(t,s)A(s)y$ for $y\in Y$, $0\leq s\leq t\leq T$.

Then we give the definition of the Yosida approximation [3].

Definition 2.1. Under the assumption (i), the family $\{A_\lambda(t)\}\subset B(X)$ is said to be the Yosida approximation for $\{A(t)\}$ if for each $t\in[0,T]$

$A_\lambda(t)=\lambda^{-1}[I-J_\lambda(t)],\quad J_\lambda(t)=(I+\lambda A(t))^{-1}\in B(X),\quad \lambda>0,\quad \lambda\beta<1.$

For $\lambda>0$, $\lambda\beta<1$, $\lambda\tilde{\beta}<1$, $A_\lambda(t)$ is defined for all $t\in[0,T]$, bounded in $B(X)$ by (i), $\tilde{A}_\lambda(t)$, the Yosida approximation for $\tilde{A}(t)$ is also defined for each t, bounded in $B(Y)$ by (ii), coincides with $A_\lambda(t)$ on Y [1; Prop. 2.3] and therefore $A_\lambda(t)$ is strongly continuous (X) by (iii).

Then the evolution operator $\{U_\lambda(t,s)\}$ for $\{A_\lambda(t)\}$ is defined uniquely by the solution of the Cauchy problem in X [4]:

(2.1) $\qquad (d/dt)u_\lambda(t,s,x)=-A_\lambda(t)u_\lambda(t,s,x),\qquad u_\lambda(s,s,x)=x,$

for $0\le s\le t\le T$, $x\in X$,

(2.2) $\qquad U_\lambda(t,s)x=u_\lambda(t,s,x).$

Now we can state the theorem to be proved.

Theorem 2.2. *Under the assumptions above there exists the evolution operator $\{U(t,s)\}$ for $\{A(t)\}$. Moreover $U_\lambda(t,s)$ converges to $U(t,s)$ strongly in $B(X)$ uniformly for t,s as $\lambda\searrow 0$.*

§ 3. Proof. The evolution operator $\{V_\lambda(t,s)\}$ for $\{A_\lambda(t_\lambda)\}$, a step function of t, is well defined as in § 2, where $t_\lambda=[t/\lambda]\cdot\lambda$. We first show the following lemma.

Lemma 3.1. *The evolution operator $V_\lambda(t,s)$ converges to some operator $U(t,s)$ strongly in $B(X)$ uniformly for t,s as $\lambda\searrow 0$ and $\{U(t,s)\}$ is the evolution operator for $\{A(t)\}$.*

Then the theorem can be proved easily. This process is necessary because we need uniform boundedness of $V_\lambda(t,s)$ in $B(Y)$ but we can tell nothing about uniform boundedness of $U_\lambda(t,s)$ in $B(Y)$ for lack of information about strong measurability of $J_\lambda(t)$ as a $B(Y)$-valued function.

Proof of Lemma. For $\lambda>0$, $\lambda\beta<1$, $\lambda\tilde{\beta}<1$, $\tilde{A}_\lambda(t_\lambda)$ is defined for all t, piecewise constant and there exists the evolution operator for it, which coincides with $\{V_\lambda(t,s)\}$ on Y by (ii).

Moreover $V_\lambda(t,s)$ satisfies the estimates:

(3.1) $\qquad \|V_\lambda(t,s)\|_X\le M\cdot\exp[\beta(t-s)/(1-\lambda\beta)],\qquad s\le t,$

(3.2) $\qquad \|V_\lambda(t,s)\|_Y\le\tilde{M}\cdot\exp[\tilde{\beta}(t-s)/(1-\lambda\tilde{\beta})],\qquad s\le t.$

In fact, $v_\lambda(t,s,x)=\exp[(t-s)/\lambda]\cdot V_\lambda(t,s)x$ satisfies the following:

(3.3) $\quad (d/dt)v_\lambda(t,s,x)=\lambda^{-1}J_\lambda(t_\lambda)v_\lambda(t,s,x),\quad v_\lambda(s,s,x)=x,\ t\neq\lambda,2\lambda,\cdots.$

The estimate (3.1) follows by virtue of the stability assumption if we express the solution of (3.3) by series; the estimate (3.2) is obtained similarly.

Next we prove that $\{V_{1/n}(t,s)y\,;\,n\in N\}$, $y\in Y$, forms a Cauchy sequence in X uniformly for t,s. To this end we consider the following equation obtained from the definition of $V_\lambda(t,s)$:

$$\begin{aligned}(3.4)\qquad &(\partial/\partial s)V_\lambda(t,s)J_\alpha(s_\alpha)V_\mu(s,r)y\\ &\qquad=V_\lambda(t,s)[A_\lambda(s_\lambda)J_\alpha(s_\alpha)-J_\alpha(s_\alpha)A_\mu(s_\mu)]\cdot V_\mu(s,r)y,\end{aligned}$$

for $s\neq k\lambda$, $s\neq k\mu$, $s\neq k\alpha$, $k=1,2,\cdots$, $y\in Y$, where $\alpha>0$, $\alpha\beta<1$, $\alpha\tilde{\beta}<1$, $s_\alpha=[s/\alpha]\cdot\alpha$ and $J_\alpha(s)=(I+\alpha A(s))^{-1}$. The parameter α is independent of λ,μ and determined later in (3.13).

Since the right hand side of (3.4) is piecewise continuous and uniformly bounded in X, we can integrate (3.4) to get

$$\begin{aligned}(3.5)\qquad &\int_r^t(\partial/\partial s)V_\lambda(t,s)J_\alpha(s_\alpha)V_\mu(s,r)yds\\ &\qquad=\int_r^t V_\lambda(t,s)[A_\lambda(s_\lambda)J_\alpha(s_\alpha)-J_\alpha(s_\alpha)A_\mu(s_\mu)]\cdot V_\mu(s,r)yds,\end{aligned}$$

for $r\leqq t$, $y\in Y$. Since $s_\alpha=[s/\alpha]\cdot\alpha$, we have from (3.5)

$$\begin{aligned}(3.6)\qquad &V_\mu(t,r)y-V_\lambda(t,r)y\\ &\qquad=\alpha[A_\alpha(t_\alpha)V_\mu(t,r)y-V_\lambda(t,r)A_\alpha(r_\alpha)y]\\ &\qquad\quad+\{V_\lambda(t,r_\alpha+\alpha)[J_\alpha(r_\alpha+\alpha)-J_\alpha(r_\alpha)]V_\mu(r_\alpha+\alpha,r)y\\ &\qquad\qquad+\cdots+V_\lambda(t,t_\alpha)[J_\alpha(t_\alpha)-J_\alpha(t_\alpha-\alpha)]\cdot V_\mu(t_\alpha,r)y\}\\ &\qquad\quad+\int_r^t V_\lambda(t,s)[A_\lambda(s_\lambda)J_\alpha(s_\alpha)-J_\alpha(s_\alpha)A_\mu(s_\mu)]V_\mu(s,r)yds,\end{aligned}$$

for $r\leq t$, $y\in Y$.

To estimate the right hand side of (3.6) we use

$$(3.7)\qquad \|J_\alpha(t+\alpha)-J_\alpha(t)\|_{Y,X}\leq\text{Const}\cdot\alpha\cdot\sup_t\|A(t+\alpha)-A(t)\|_{Y,X},$$

$$\begin{aligned}(3.8)\qquad &\|A_\lambda(s_\lambda)J_\alpha(s_\alpha)-J_\alpha(s_\alpha)A_\mu(s_\mu)\|_{Y,X}\\ &\qquad\leq\text{const}\cdot\left[\frac{\lambda+\mu}{\alpha}+\sup_{|t'-t|\leq\alpha+\lambda+\mu}\|A(t')-A(t)\|_{Y,X}\right].\end{aligned}$$

The proof of (3.7) is easy, so omitted. To prove (3.8) we notice the decomposition

$$\begin{aligned}(3.9)\qquad &A_\lambda(s_\lambda)J_\alpha(s_\alpha)-J_\alpha(s_\alpha)A_\mu(s_\mu)\\ &\qquad=[A_\lambda(s_\lambda)-A(s_\alpha)J_\lambda(s_\alpha)]J_\alpha(s_\alpha)+[A(s_\alpha)J_\lambda(s_\alpha)J_\alpha(s_\alpha)\\ &\qquad\quad-J_\alpha(s_\alpha)A(s_\alpha)J_\mu(s_\alpha)]+J_\alpha(s_\alpha)[A(s_\alpha)J_\mu(s_\alpha)-A_\mu(s_\mu)].\end{aligned}$$

Then (3.8) can be obtained with the aid of the estimates

$$(3.10)\qquad \|A_\lambda(s_\lambda)-A(s_\alpha)J_\lambda(s_\alpha)\|_{Y,X}\leq\text{Const}\cdot\sup_{|t'-t|\leq\lambda+\alpha}\|A(t')-A(t)\|_{Y,X},$$

$$(3.11)\qquad \|A(s_\alpha)J_\lambda(s_\alpha)J_\alpha(s_\alpha)-J_\alpha(s_\alpha)A(s_\alpha)J_\mu(s_\alpha)\|_{Y,X}\leq\text{Const}\cdot\frac{\lambda+\mu}{\alpha},$$

$$(3.12)\qquad \|A(s_\alpha)J_\mu(s_\alpha)-A_\mu(s_\mu)\|_{Y,X}\leq\text{Const}\cdot\sup_{|t'-t|\leq\alpha+\mu}\|A(t')-A(t)\|_{Y,X}.$$

Hence by (3.7), (3.8) and uniform boundedness of $V_\lambda(t,s)$ in $B(X)$ and $B(Y)$ (see (3.1), (3.2)), we get from (3.6):

$$\begin{aligned}(3.13)\qquad &\|V_\mu(t,r)y-V_\lambda(t,r)y\|\\ &\qquad\leq\text{Const}\cdot\left[\alpha+\frac{\lambda+\mu}{\alpha}+\sup_{|t'-t|\leq\alpha+\lambda+\mu}\|A(t')-A(t)\|_{Y,X}\right]\cdot\|y\|_Y.\end{aligned}$$

This means that $\{V_{1/n}(t,r)y\}_{n\in N}$ forms a Cauchy sequence in X uniformly for t,r. Since Y is dense in X and $V_\lambda(t,r)$ is uniformly bounded in $B(X)$ for λ, t, r, we conclude that $V_\lambda(t, r)x$ converges strongly in X to, say, $U(t, r)x$ as $\lambda \searrow 0$ uniformly in t, r for each $x \in X$ and $U(t, r)x$ is strongly continuous in X.

The conditions (a) and (b) of the evolution operator can be obtained form the corresponding relations of $V_\lambda(t, s)$ by passing to the limit or limit infimum as $\lambda \searrow 0$. The condition (c) is also obtained if we notice the relation

$$h^{-1}[V_\lambda(t+h, t)y - y] = -h^{-1}\int_t^{t+h} V_\lambda(t+h, s)A_\lambda(s_\lambda)y\,ds, \qquad h>0,\ y \in Y,$$

pass to the limit $\lambda \searrow 0$, and use continuity of $U(t, s), A(t)$. The proof of (d) and uniqueness of the evolution operator is the same as that of Kato [1], so we may omit it.

Proof of the theorem. It suffices to prove that the difference $U_\lambda(t, r)y - V_\lambda(t, r)y$ converges to zero in X as $\lambda \searrow 0$ uniformly in t, r for each $y \in Y$. This follows from the relation

$$V_\lambda(t, r)y - U_\lambda(t, r)y = \int_r^t U_\lambda(t, s)[A_\lambda(s) - A_\lambda(s_\lambda)]V_\lambda(s, r)y\,ds.$$

§4. Remarks to "quasi-stable" case. In the quasi-stable case, $\beta, \tilde{\beta}$ are Lebesgue upper integrable functions of t[2; p. 648]. Our method also applies to this case, but we need more care about choice of $A_n(t_n)$, $n \in N$, and $J_p(t_p)$, $p \in N$, where $A_n(t_n)$, $J_p(t_p)$ correspond to $A_\lambda(t_\lambda)$, $J_\alpha(t_\alpha)$ in the stable case. We can assume without loss of generality that $\beta, \tilde{\beta}$ are Lebesgue integrable and greater than a positive constant a.e., if necessary, replacing them by dominating integrable functions. Then we choose the Yosida approximation $A_n(t_n)$ as follows:

$$A_n(t_n) = A(t_n)J_n(t_n), \qquad J_n(t) = \left(I + \frac{1}{n^2\bar{\beta}(t)}A(t)\right)^{-1} \qquad \text{a.e., } n \in N,$$

where $\bar{\beta}(t) = \max\{\beta(t), \tilde{\beta}(t)\}$. The step function t_n of t must be chosen so that $\bar{\beta}(t_n) \to \bar{\beta}(t)$ as $n \nearrow \infty$ in L^1 [2; p. 651]. The details of the proof may be omitted.

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