# 2. Global Solutions of the Boltzmann Equation in a Bounded Convex Domain

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1. **Introduction.** We consider the Boltzmann equation

$$\frac{\partial F}{\partial t}+\sum_{i=1}^{3}\xi_i\frac{\partial F}{\partial x_i}=J(F,F), \tag{1}$$

which describes the change in time of the distribution function of the arguments space $x$ and velocity $\xi$. Here $J(F,F)$ is the collision integral [1]. The equilibrium solution of (1) is $F=\omega$, where

$$\omega(\xi)=\frac{1}{(2\pi)^{3/2}}\exp\left(-\frac{1}{2}|\xi|^2\right).$$

As we are interested in solutions of (1) which are close to $F=\omega$, we introduce $f(x,\xi)$ by

$$F=\omega+\omega^{1/2}f. \tag{2}$$

Then the equation satisfied by $f$ is

$$\frac{\partial f}{\partial t}=Bf+\Lambda\Gamma(f,f). \tag{3}$$

The explicit form of the operator $B$ is

$$\begin{aligned}(Bf)(x,\xi)=&-\sum_{i=1}^{3}\xi_i\frac{\partial f(x,\xi)}{\partial x_i}-\nu(\xi)f(x,\xi)\\&+\int_{R^3}K(\xi,\eta)f(x,\eta)d\eta,\end{aligned} \tag{4}$$

where $\nu(\xi)$, the collision frequency, is a certain unbounded positive function of $\xi$ and $K(\xi,\eta)$, the collision kernel, is a symmetric function of $\xi$ and $\eta$. The operator $\Lambda$ is the multiplication operator by $\nu(\xi)$ and $\Gamma(f,f)$ denotes the quadratic term. Note that $J(\omega,\omega)=0$. We shall use Grad's estimates [1], [2] for $\nu(\xi)$, $K(\xi,\eta)$ and $\Gamma(f,f)$ in computations. This means that the potential is a hard potential in the sense of Grad and that the angular cut-off assumption is made for the differential cross section. A typical example satisfying these conditions is a gas of rigid spheres. The initial value problems for the Boltzmann equation on the torus and on the entire space have been studied earlier in [4] and [5], respectively. In this note, we treat the initial boundary value problem for the case of specular reflection boundary condition. Our

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aim is to show the existence of solutions in the large for the initial data near equilibrium.

**2. Decay estimates.** Let us consider a bounded convex domain $\Omega$ in $R^3$ and assume that the boundary $\partial\Omega$ is three times continuously differentiable. In addition, the principal curvatures are assumed to be positive on $\partial\Omega$. The appropriate function space is $S_\alpha$, $\alpha\geqq 0$, i.e., the set of all functions satisfying

(i) $f$ is a continuous function on $\bar{\Omega}\times R^3$,

(ii) for $(x,\xi)\in\partial\Omega\times R^3$,

$$f(x,\xi)=f(x,\xi-2n_x(\xi\cdot n_x)),$$

where $n_x$ denotes the inner normal to $\partial\Omega$ at $x$,

(iii) $\sup_x (1+|\xi|^2)^{\alpha/2}|f(x,\xi)|\to 0, \qquad \text{as } |\xi|\to\infty.$

On this space we have the norm

$$\|f\|_\alpha=\sup_{x,\xi}(1+|\xi|^2)^{\alpha/2}\cdot|f(x,\xi)|.$$

Taking into account of the specular reflection boundary condition, we see that the operator $B$ generates a bounded semi-group $\{V(t)\}$ in $S_\alpha$ for any $\alpha\geqq 0$. The imaginary axis belongs to the resolvent set of $B$ except for $\lambda=0$, which is an isolated eigenvalue of $B$. The resolvent $(\lambda-B)^{-1}$ has a simple pole at $\lambda=0$. The residue of the resolvent at $\lambda=0$ is a projection operator $P$ of finite rank $r$, $2\leqq r\leqq 5$. By using a theorem of Jörgens and Vidav, we obtain the following estimate.

**Theorem 1.** *For any $\gamma>0$ small enough, there exists a constant $M>0$ depending only on $\alpha$ and $\gamma$ such that*

$$\|V(t)(I-P)\|_{S_\alpha\to S_\alpha}\leqq Me^{-\gamma t}, \qquad \text{for } t\geqq 0. \tag{5}$$

**3. Global solutions.** The space $X_{\alpha,\gamma}$ is the set of functions of argument $t$ with values in $S_\alpha$ satisfying

(i) $f$ is a continuous function on $[0,\infty)$,

(ii) $\sup_t e^{\gamma t}\|f(t)\|_\alpha<\infty$.

$X_{\alpha,\gamma}$ is endowed with the norm

$$\|f\|_{\alpha,\gamma}=\sup_t e^{\gamma t}\|f(t)\|_\alpha.$$

We denote by $N_\alpha$ the set of all functions $f\in S_\alpha$ satisfying $Pf=0$. This is equivalent to saying that $f\in N_\alpha$ if and only if

$$\iint_{\Omega\times R^3} f(x,\xi)\psi_i(x,\xi)dxd\xi=0, \qquad i=1,2,\cdots,r,$$

where $\{\psi_i\}$ is a basis of the nullspace of $B$. $Y_{\alpha,\gamma}$ denotes the set of all functions $f\in X_{\alpha,\gamma}$ taking its values in $N_\alpha$. Now we consider the integral equation

$$f(t)=V(t)\phi+\int_0^t V(t-s)\Lambda\Gamma(f(s),f(s))ds, \tag{6}$$

which is derived formally from (3) with $f(0)=\phi$. Note that the integral in the right side of (6) is well defined in $S_{\alpha-1}$ for any continuous function $f$ with values $f(t)$ in $S_\alpha$, $\alpha\geqq 1$.

**Theorem 2.** *If $\gamma > 0$ is small enough and $\alpha \geqq 1$, there exists a positive constant d depending only on $\alpha$ and $\gamma$ such that, for any $\phi \in N_\alpha$ with $\|\phi\|_\alpha \leqq d^2$,* (6) *has a unique solution $f \in Y_{\alpha,\gamma}$ with $\|f\|_{\alpha,\gamma} \leqq d$. The mapping $\phi \rightarrow f$ is continuous and indefinitely differentiable. Furthermore, $f = f(t, x, \xi)$ satisfies*

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + \sum_{i=1}^{3} \xi_i \frac{\partial}{\partial x_i}\right] f(t, x, \xi) \\ (7) \qquad &\quad = -\nu(\xi) f(t, x, \xi) + \int K(\xi, \eta) f(t, x, \eta) d\eta \\ &\quad + \nu(\xi)(\Gamma(f(t), f(t)))(x, \xi), \end{aligned}$$

*pointwise on $(0, \infty) \times \Omega \times R^3$. Here $[\partial/\partial t + \sum_{i=1}^{3} \xi_i \partial/\partial x_i]$ means the differentiation in the direction $(1, \xi_1, \xi_2, \xi_3)$ for every fixed $\xi$.*

The proof is based on Theorem 1 and the implicit function theorem. A similar result has been obtained by Guiraud [3] for the case of pseudo reflection boundary condition.

## References


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