59. *The Behavior of Solutions of the Equation of Kolmogorov-Petrovsky-Piskunov*

By Kôhei UCHIYAMA
Tokyo Institute of Technology

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1. Given a source function $F(u)$ on $[0,1]$ which is positive on $0<u<1$ with $F(0)=F(1)=0$, continuously differentiable on $0\leqq u\leqq 1$ and $F'(0)>0$, let us consider the Cauchy problem

$$(1)\qquad \frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+F(u)\qquad t>0,\ x\in R=(\infty,+\infty)$$
$$\lim_{t\downarrow 0} u(t,x)=f(x),$$

where an initial function f is piecewise continuous on R with $0\leqq f\leqq 1$.

Let w_c denote a propagating front associated with speed c: $w_c(x-ct)$ is a non-trivial solution of (1)*) ($0\leqq w_c\leqq 1$), with normalization $w_c(0)=1/2$. Our interest in this article lies in such phenomena that

(2) $\qquad u(t,x+m(t))$ *converges to* $w_c(x)$ *as* $t\to\infty$,

where

$$m(t)=\sup\left\{x\,;\,u(t,x)=\frac{1}{2}\right\}.$$

If $f\not\equiv 0$ and $f(x)\to 0$ as $x\to\infty$, we have that $u(t,x)\to 0$ as $x\to\infty$ and $u(t,x)\to 1$ as $t\to\infty$ (cf. [1]) and in particular that $m(t)$ has a definite value for large t. Aronson and Weinberger proved in [1] that there is a positive constant c_0 called the minimal speed such that the propagating front associated with speed c exists iff $c^2\geqq c_0^2$ ($c_0^2\geqq 2F'(0)$; the propagating front is unique up to the translation for each c) (cf. also [3]). Such a phenomenon as described in (2) was first observed by Kolmogorov, Petrovsky and Piskunov [3]: they proved that (3) holds with $c=c_0$ if we take $f=I_{(-\infty,0)}$ (the indicator function of the negative real axis). Kametaka [2] proved (2) when f belongs to a certain class of monotone functions. These are improved in the theorems of the next section which confirm that (2) is valid to a wide class of initial functions that contains all $f(0\leqq f\leqq 1)$ with non-empty compact support.

2. Let $A(x)$ be a positive function on R such that $A(x+x_0)\sim A(x)$ as $x\to\infty$ for each $x_0\in R$. We will assume one of the following conditions on the behavior of f for large positive x:

(3) $\qquad f(x)=0\qquad$ for $x>N_1(N_1\in R)$ and $\quad f\not\equiv 0$

or

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*) Trivial solutions are $u\equiv 0$ and $u\equiv 1$.

(4) $f(x) \sim A(x)e^{-bx}$ as $x \to \infty$ $(b>0)$.

We must further impose a slight (probably technical) restriction on the tail of f at negative infinity:

(5) f is non-decreasing for $x<N_2$ $(N_2 \in R)$ or $\liminf_{x\downarrow -\infty} f(x)>0$.

Theorem 1. *Let f $(0 \leqq f \leqq 1)$ satisfy the condition* (3) *or the condition* (4) *with $b>c_0-\sqrt{c_0^2-2F'(0)}$ and satisfy the condition* (5). *Then* (2) *holds with $c=c_0$ uniformly in $x>N$ for each $N \in R$.**)

Theorem 2. *Let f $(0 \leqq f \leqq 1)$ satisfy the condition* (4) *with $0<b \leqq c_0-\sqrt{c_0^2-2F'(0)}$ and the condition* (5). *Then* (2) *holds with $c=b/2+F'(0)/b$ uniformly in $x>N$ for each $N \in R$.*

The next theorem is complementary to these theorems.

Theorem 3. *Let f $(0 \leqq f \leqq 1)$ be differentiable and positive and satisfy $\limsup_{x\uparrow\infty}[-f'(x)/f(x)] \leqq 0$ and $\lim_{x\uparrow\infty} f(x)=0$. Then, under the condition* (5), *$\lim_{t\uparrow\infty} u(t, x+m(t))=1/2$ uniformly on each compact set of R.*

The method of the proofs of Theorems 1 and 2 is similar to that used by the authors mentioned in the previous section and summarized as follows. Define

$$M(t)=\sup\left\{u(t,x)\,;\ \frac{\partial u}{\partial x}(t,y)<0 \text{ for all } y>x\right\}.$$

Then under assumptions of Theorems 1 or 2 we have $M(t)\to 1$ as $t\to\infty$. Set

$$\phi(t,w)=\frac{\partial u}{\partial x}(t,x(t,w)) \qquad 0 \leqq w \leqq M(t)$$

where $x(t,\cdot)$ is the inverse function of $u(t,\cdot)$. Considering ϕ as a functional of f, we denote it by $\phi(t,w\,;f)$. Then $\phi(t,w\,;w_c)$ is independent of t, since $w_c(x-ct)$ solves (1). We set $\tau_c(w)=\phi(t,w\,;w_c)$. The theorems are proved by showing that $\phi(t,w\,;f)$ converges to $\tau_c(w)$ as $t\to\infty$. This is carried out, at first, for an appropriately chosen initial function, say f_*, which is subject to several restrictions, and then for general f by applying a comparison theorem based on the maximum principle of the parabolic equation to the equation

$$\frac{\partial w}{\partial t}=\frac{1}{2}\psi^2\frac{\partial^2\omega}{\partial w^2}-F(w)\frac{\partial\omega}{\partial w}+\left[F'(w)+\frac{1}{2}(\phi+\psi)\frac{\partial^2\phi}{\partial w^2}\right]\omega,$$

where $\phi=\phi(t,w\,;f)$, $\psi=\phi(t,w\,;f_*)$ and $\omega=\phi-\psi$. This equation has the singularity at $w=0$, but we can justify the application of the comparison theorem using estimates: $\phi(t,w)=O(\sqrt{|\log\omega|}\,w)$ and $(\partial^2\phi/\partial w^2)(t,w)=o(\sqrt{|\log w|}/w)$ as $w\downarrow 0$ uniformly in $T^{-1}<t<T$, which follow from Conditions (3) or (4).

3. Here are four theorems: the first one maintains the strong stability (in a certain sense) of the front w_c when $c^2>2\sup_{0<u<1}F'(u)$

*) The theorem is valid also in case $F'(0)=0$ if we adopt the latter one in the condition (5).

and the remaining three give the criterion of whether or not $m(t)$ can be replaced by $ct+\text{const}$ in Theorems 1 or 2.

Theorem 4. *Let* $(1/2)c^2 \geqq \gamma = \sup_{0<u<1} F'(u)$ *and* $f(x)=w_c(x+x_0)+O(e^{-bx})$ *with some constants* b *and* x_0. *Then*

$$u(t, x+ct)=w_c(x+x_0)+O(e^{-pt-bx}),$$

where $p=b(c-b/2-\gamma/b)$ (*assuming* $c>0$, $p>0$ *is equivalent to* $c-\sqrt{c^2-2\gamma}<b<c+\sqrt{c^2-2\gamma}$), *and if* $b>c$ *we have*

$$u(t, x+ct)=w_c(x+x_0)+O\left(\frac{1}{\sqrt{t}} e^{-(c^2/2-\gamma)t}\right)$$

uniformly in $x>N$ *for each* $N \in R$.

Theorem 5. *Let* $c>c_0$. *Suppose that there exists* $\lim_{x\uparrow\infty} e^{bx}f(x)=a\leqq\infty$ *where* $b=c-\sqrt{c^2-2F'(0)}$ *and that*

$$\int_{0+} |F'(0)-F'(u)|\, u^{-1}du < \infty.$$

Then $m(t)-ct$ *is bounded iff* $0<a<\infty$. *If this is the case, under the condition* (5), *it holds that with some constant* x_0

(6) $$u(t, x+ct) \longrightarrow w_c(x+x_0) \qquad \text{as } t\to\infty$$

uniformly in $x>N$ *for each* $N\in R$.

Theorem 6. *Let*

$$\int_{0+} |F'(0)-F'(u)|\,|\log u|\, u^{-1}du<\infty.$$

Assume that there exists another source function F^* *such that* $F^*\geqq F$, $F^*\not\equiv F$ *and the minimal speed* c_0 *is common to* F *and* F^* (*this implies* $c_0^2=2F'(0)$). *Further assume that there exists* $\lim_{x\to\infty} e^{c_0x}f(x)/x=a\leqq\infty$. *Then we have the same conclusion as Theorem* 5 *where* c *is replaced by* c_0.

Theorem 7. *Let* $c_0^2>2F'(0)$. *Then* $m(t)-c_0t$ *is bounded, provided that* $f\not\equiv 0$ *and* $\lim_{x\uparrow\infty} e^{bx}f(x)=0$ *for some constant* $b>c_0-\sqrt{c_0^2-2F'(0)}$. *In particular* (6) *holds with* $c=c_0$ *under the assumptions of Theorem* 1.

4. Kolmogorov *et al.* [3] showed that $m^{\cdot}(t)=dm(t)/dt\to c_0$ as $t\to\infty$ in case $f=I_{(-\infty,0)}$. The next theorem generalizes the result.

Theorem 8. *Suppose that for some continuous function* $k(t)$ *there exists* $\lim_{t\uparrow\infty} u(t, x+k(t))=g(x)$ *in locally* L_1 *sense, where* g *is not a constant. Then* $g(x)=w_c(x+x_0)$ *with some constants* x_0 *and* c, $c^2\geqq c_0^2$. *If* $m(t)$ *is defined (for large* t*) by* $u(t, m(t))=1/2$ *and* $m(t)-k(t)$ *being bounded, then* $m^{\cdot}(t)\to c$ *as* $t\to\infty$. *Furthermore* $v(t, x)=u(t, x+m(t))$, $\partial v/\partial x$ *and* $\partial^2 v/\partial x^2$ *converge to* w_c, w_c' *and* w_c'', *respectively, as* $t\to\infty$ *locally uniformly*.

If $F(u)\leqq F'(0)u$ for all $0\leqq u\leqq 1$, we can obtain a fine estimate of $m(t)$, which is an improvement of McKean [4].

Theorem 9. *Suppose* $F(u)\leqq F'(0)u$ *for all* $0\leqq u\leqq 1$ *and*

$$\int_{0+} |F'(0)-F'(u)|\,|\log u|\, u^{-1}du<\infty.$$

Let f satisfy the condition (3). *Then*

$$\text{const} \leqq m(t) - c_0 t + \frac{3 \log t}{2c_0} \leqq O(\log \log t).$$

The proofs of these theorems will be published elsewhere.

References

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