46. *Jackson-Type Estimates for Monotone Approximation*

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1. Denote by H_n, the set of all polynomials of degree n or less, and by $H_{n,k}(k\leqq n)$, the set of all $P_n\in H_n$ satisfying $P_n^{(k)}(x)\geqq 0$ on $[0,1]$. Define $E_n(f)=\inf_{P_n\in H_n}\|f-P_n\|$ and $E_{n,k}(f)=\inf_{P_n\in H_{n,k}}\|f-P_n\|$, where $\|\cdot\|$ is the supremum norm of functions continuous on $[0,1]$.

Many authors have investigated on the degree of monotone approximation. For instance, see [1]–[4], [6]–[9]. We would like to prove the inequality

$$E_{n,k}(f)\leqq\frac{C}{n^k}E_{n-k}(f^{(k)}) \tag{1}$$

for $f\in C^k[0,1]$. Here C denotes a positive constant depending upon k. This result is true for the unconstrained degree of approximation $E_n(f)$.

For the function x^{2n+1} $(n=1,2,\cdots)$ that increases on $[-1,+1]$, R. A. DeVore [1] proved the following: Let $\alpha=\log_3 4-1$ and $\beta=1-\log_a 2$, with $a=2+3^{1/2}$. Then there exist constants $C_1,C_2>0$, such that

$$C_1n^{\alpha}2^{-2n}\leqq E_{2n,1}(x^{2n+1})\leqq C_2n^{\beta}2^{-2n},\qquad n=1,2,\cdots.$$

However we have

$$E_{2n-1}(x^{2n})=\|x^{2n}-(x^{2n}-2^{-(2n-1)}C_{2n}(x))\|=2^{-(2n-1)},$$

where $C_{2n}(x)$ is the Chebyshev polynomial of degree $2n$. Hence, in this case, (1) does not hold for $k=1$. For x^{2n+1} vanishes at $x=0$.

J. A. Roulier [6] examined this problem and proved the following:

(i) For $f\in C^1[0,1]$ with $f'(x)>0$ on $[0,1]$, we have

$$E_{n,1}(f)\leqq\frac{5}{2n^{1/2}}E_{n-1}(f'),\qquad n\geqq N(f).$$

(ii) For any $k=2,3,\cdots$ and $f\in C^k[0,1]$ with $f^{(k)}(x)>0$ on $[0,1]$, we have

$$E_{n,k}(f)\leqq\frac{2}{n}E_{n-k}(f^{(k)}),\qquad n\geqq N(f,k),$$

$N(f,k)$ denoting a certain positive integer depending upon f and k.

The purpose of this paper is to prove that the inequality (1) holds under the assumption of Roulier.

2. **Theorem.** *For any $k=1,2,\cdots$ and $f\in C^k[0,1]$ satisfying $f^{(k)}(x)>0$ on $[0,1]$, we have*

$$E_{n,k}(f) \leqq \frac{C}{n^k} E_{n-k}(f^{(k)}), \qquad n \geqq N(f, k),$$

with C depending only on k.

Proof. Let $Q_{n-k}(x)$ be the polynomial of best approximation from H_{n-k} to $f^{(k)}(x)$ on $[0,1]$. For a fixed integer n ($n \geqq k$), we define

$$\phi(x) = \int_0^x \int_0^{x_1} \cdots \int_0^{x_{k-1}} Q_{n-k}(t)\,dt\,dx_{k-1} \cdots dx_1 - f(x).$$

Because of $\phi^{(k)}(x) = Q_{n-k}(x) - f^{(k)}(x) \in C[0,1]$, and by using the results of Trigub [10] (see also Malozemov [5]), we see that there exists a $P_n \in H_n$ with the properties

$$\|\phi^{(r)} - P_n^{(r)}\| \leqq \frac{C_1}{n^{k-r}} \omega\left(\phi^{(k)}, \frac{1}{n}\right), \qquad r = 0, 1, \cdots, k,$$

where C_1 is a constant depending only on k, and $\omega(g, \cdot)$ is the modulus of continuity. Putting $r=0$, we get

$$\begin{aligned} \|\phi - P_n\| &\leqq \frac{C_1}{n^k} \omega\left(\phi^{(k)}, \frac{1}{n}\right) \\ &\leqq \frac{2C_1}{n^k} E_{n-k}(f^{(k)}) = \frac{C}{n^k} E_{n-k}(f^{(k)}). \end{aligned}$$

When $r=k$, we obtain for $0 \leqq x \leqq 1$

$$\begin{aligned} P_n^{(k)}(x) &\leqq \phi^{(k)}(x) + C_1 \omega\left(\phi^{(k)}, \frac{1}{n}\right) \\ &= Q_{n-k}(x) - f^{(k)}(x) + C_1 \omega\left(\phi^{(k)}, \frac{1}{n}\right) \qquad (2) \\ &\leqq Q_{n-k}(x) - f^{(k)}(x) + C E_{n-k}(f^{(k)}). \end{aligned}$$

Define

$$\tilde{P}_n(x) = \int_0^x \int_0^{x_1} \cdots \int_0^{x_{k-1}} Q_{n-k}(t)\,dt\,dx_{k-1} \cdots dx_1 - P_n(x).$$

Thus, we have a sequence of polynomials $\tilde{P}_n(x) \in H_n$ such that

$$\begin{aligned} |f(x) - \tilde{P}_n(x)| &= |\phi(x) - P_n(x)| \\ &\leqq \frac{C}{n^k} E_{n-k}(f^{(k)}) \qquad \text{on } [0,1]. \end{aligned}$$

Further, using (2) and $f^{(k)}(x) > 0$ on $[0,1]$, we have for $0 \leqq x \leqq 1$

$$\begin{aligned} \tilde{P}_n^{(k)}(x) &= Q_{n-k}(x) - P_n^{(k)}(x) \\ &\geqq f^{(k)}(x) - C E_{n-k}(f^{(k)}). \end{aligned}$$

By the polynomial approximation theorem of Weierstrass, the right hand term is $\geqq 0$ for $n \geqq N(f, k)$. This completes the proof.

3. The case $k=1$ in the theorem follows from the stronger inequality

$$E_{n,1}(f) \leqq C E_n(f), \qquad n \geqq N(f),$$

which was shown by J. A. Roulier [7].

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