6. *On the Homogeneous Linear Systems of Differential Equations with Variable Coefficients*

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1. Main Theorem. Consider the homogeneous linear system of differential equations

(1.1) $$\dot{x}=A(t)x \qquad (x: n\text{-vector})$$

where the coefficient $A(t)$ is a continuously differentiable $n\times n$ matrix function defined in an interval I in R^1 $(0\in I)$. In this paper the equation (1.1) is said to be reduced to another equation

(1.2) $$\dot{y}=B(t)y$$

of the same form under the transformation

(1.3) $$x=e^{St}y$$

if there exists a constant $n\times n$ matrix S such that the transformation (1.3) transforms (1.1) into (1.2).

Now, we show the necessary and sufficient condition for the equation (1.1) is reduced to (1.2) under the transformation (1.3), and also its some examples and some applications to solve the equation (1.1) explicitly.

Theorem 1. *The homogeneous linear system of differential equations* (1.1) $\dot{x}=A(t)x$ *with a continuously differentiable coefficient matrix* $A(t)$ *is reduced to the homogeneous linear system* (1.3): $\dot{y}=B(t)y$ *of the same form, if and only if there exists a constant matrix* S *such that the equations*

(1.4) $$\dot{A}(t)=SA(t)-A(t)S-e^{St}\dot{B}(t)e^{-St}$$

(1.5) $$A(0)=S+B(0)$$

hold.

Proof. Under the transformation $x=e^{St}y$,

$$\dot{x}=e^{St}Sy+e^{St}\dot{y}=A(t)e^{St}y$$

and therefore (1.1) is transformed into the equation

$$\dot{y}=\{e^{-St}A(t)e^{St}-S\}y.$$

Now (1.1) is reduced to (1.2) under the transformation (1.3) if and only if the following equality holds:

(1.6) $$B(t)=e^{-St}A(t)e^{St}-S \qquad (t\in I).$$

This and (1.5) are equivalent to

(1.7) $$e^{St}\frac{d}{dt}(e^{-St}A(t)e^{St})\cdot e^{-St}=-e^{st}\dot{B}(t)e^{-St}$$

(1.8) $$B(0)=A(0)-S,$$

and (1.7) is equivalent to the equation (1.4). Q.E.D.

Corollary 1. *The homogeneous linear system of differential equations* (1.1): $\dot{x}=A(t)x$ *with a continuously differentiable coefficient* $A(t)$ *is reduced to*

$$\dot{y}=By \tag{1.2$'$}$$

of the same form with constant coefficient B *under the transformation* (1.3) $x=e^{St}y$, *if and only if there exists a constant* $n\times n$ *matrix* S *satisfying the equations*

$$\dot{A}(t)=SA(t)-A(t)S \tag{1.4$'$}$$

$$A(0)=S+B. \tag{1.5$'$}$$

Proof is obvious from Theorem 1.

Remark 1. Corollary 1 shows that, if the equation (1.1) is reduced to (1.2)$'$ under the transformation (1.3), the fundamental matrix solution $\Phi(t)$ of (1.1) satisfying $\Phi(0)=I$ is given by the following form:

$$\Phi(t)=e^{St}e^{Bt}. \tag{1.9}$$

Note that S and B are not commutative except for special cases.

2. Examples. Example 1. Consider the equation (1.1) with the coefficient matrix

$$A(t)=\begin{pmatrix}-1 & e^{2t}\\ 0 & -1\end{pmatrix}. \tag{2.1}$$

It can be seen easily that

$$S=\begin{pmatrix}2 & 1\\ 0 & 0\end{pmatrix},\qquad B=\begin{pmatrix}-3 & 0\\ 0 & 1\end{pmatrix} \tag{2.2}$$

satisfy the equations (1.4)$'$ and (1.5)$'$. Thus the fundamental matrix solution of (1.1) satisfying $\Phi(0)=I$ is given by (1.9), that is

$$\Phi(t)=\begin{pmatrix}e^{-t} & (1/2)(e^{t}-e^{-t})\\ 0 & e^{-t}\end{pmatrix}. \tag{2.3}$$

Example 2. Consider the equation (1.1) with the coefficient matrix

$$A(t)=\begin{pmatrix}a+c\cos 2\omega t+d\sin 2\omega t & b+d\cos 2\omega t-c\sin 2\omega t\\ -b+d\cos 2\omega t-c\sin 2\omega t & a-c\cos 2\omega t-d\sin 2\omega t\end{pmatrix}, \tag{2.4}$$

where a, b, c, d and ω are real constants. It can be seen that the matrices

$$S=\begin{pmatrix}0 & \omega\\ -\omega & 0\end{pmatrix},\qquad B=\begin{pmatrix}a+c & b+d-\omega\\ -b+d+\omega & a-c\end{pmatrix} \tag{2.5}$$

satisfy the equations (1.4)$'$ and (1.5)$'$. Hence the fundamental matrix solution $\Phi(t)$ of (1.1) with (2.4) satisfying $\Phi(0)=I$ is given by

$$\begin{aligned}\Phi(t)&=e^{St}\cdot e^{Bt}=\frac{1}{\lambda_1-\lambda_2}\begin{pmatrix}\cos\omega t & \sin\omega t\\ -\sin\omega t & \cos\omega t\end{pmatrix}\\ &\times\begin{pmatrix}(\alpha-\lambda_2)e^{\lambda_1 t}+(\lambda_1-\alpha)e^{\lambda_2 t} & \beta(e^{\lambda_1 t}-e^{\lambda_2 t})\\ \gamma(e^{\lambda_1 t}-e^{\lambda_2 t}) & (\delta-\lambda_2)e^{\lambda_1 t}+(\lambda_1-\delta)e^{\lambda_2 t}\end{pmatrix}\quad (\text{if } \lambda_1\neq\lambda_2),\end{aligned} \tag{2.6}$$

where λ_1, λ_2 are the characteristic roots of B, and $\alpha=a+c$, $\beta=b+d-\omega$, $\gamma=-b+d+\omega$ and $\delta=a-c$. In the case of $\lambda_1=\lambda_2$, we may have $\Phi(t)$ as the limit matrix of (2.6) with $\lambda_2\to\lambda_1$.

Writing $\Phi(t)=(\phi_{ij}(t))$, we have more precisely

$$\begin{aligned}
\phi_{11}(t)&=(\lambda_1-\lambda_2)^{-1}[e^{\lambda_1 t}\{(\alpha-\lambda_2)\cos\omega t+\gamma\sin\omega t\}\\
&\qquad+e^{\lambda_2 t}\{(\lambda_1-\alpha)\cos\omega t-\gamma\sin\omega t\}]\\
\phi_{12}(t)&=(\lambda_1-\lambda_2)^{-1}[e^{\lambda_1 t}\{\beta\cos\omega t+(\delta-\lambda_2)\sin\omega t\}\\
&\qquad+e^{\lambda_2 t}\{-\beta\cos\omega t+(\lambda_1-\delta)\sin\omega t\}]\\
\phi_{21}(t)&=(\lambda_1-\lambda_2)^{-1}[e^{\lambda_1 t}\{(\lambda_2-\alpha)\sin\omega t+\gamma\cos\omega t\}\\
&\qquad+e^{\lambda_2 t}\{(\alpha-\lambda_1)\sin\omega t-\gamma\cos\omega t\}]\\
\phi_{22}(t)&=(\lambda_1-\lambda_2)^{-1}[e^{\lambda_1 t}\{(\delta-\lambda_2)\cos\omega t-\beta\sin\omega t\}\\
&\qquad+e^{\lambda_2 t}\{(\lambda_1-\delta)\cos\omega t-\beta\sin\omega t\}].
\end{aligned}\tag{2.7}$$

Remark 2. Let $2a=-11$, $b=d=6$, $2c=-9$ and $\omega=6$, then Example 2 reduces to the well known example due to Vinogradov [4].

Remark 3. Let $4a=-1$, $b=-1$, $4c=3$, $d=0$ and $\omega=1$, then Example 2 reduces to the well known example given by Markus-Yamabe [2].

Theorem 2. *The zero solution of the homogeneous linear system of differential equations* (1.1) *with the coefficient matrix of the form* (2.4) *is uniform-asymptotically stable if and only if the real parts of the characteristic roots of the matrix B given in* (2.5) *are all negative. That is*

$$\mathrm{Re}\,\{a\pm\sqrt{c^2+d^2-b^2-\omega(\omega-2b)}\}<0.$$

Example 3. Consider the equation (1.1) with the coefficient matrix

$$A(t)=\begin{pmatrix}-a(t)+(1/2)\sin 2t & 1+(1/2)(1+\cos 2t)\\ -1-(1/2)(1-\cos 2t) & -a(t)-(1/2)\sin 2t\end{pmatrix}.\tag{2.8}$$

It can be seen easily that the matrices

$$S=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\qquad B(t)=\begin{pmatrix}-a(t) & 0\\ 0 & -a(t)\end{pmatrix}\tag{2.9}$$

satisfy the equations (1.4) and (1.5). Thus the fundamental matrix solution $\Phi(t)$ of (1.1) with coefficient (2.8) is given by

$$\Phi(t)=e^{\int_0^t a(\tau)d\tau}\cdot\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}.\tag{2.10}$$

3. Application. In this section we shall give an application of Theorem 1 to (1.1) with a 2×2 matrix coefficient.

Theorem 3. *The equation* (1.1) *with a continuously differentiable 2×2 matrix coefficient*

$$A(t)=\begin{pmatrix}a(t) & b(t)\\ c(t) & d(t)\end{pmatrix}\tag{3.1}$$

is reduced to the equation (1.2) *with the coefficient*

$$B(t)=\begin{pmatrix}\phi(t) & k\\ 0 & \psi(t)\end{pmatrix}\qquad (k\text{: constant})\tag{3.2}$$

under the transformation (1.3) *with*

$$S=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\tag{3.3}$$

if the following conditions are satisfied:

$$\begin{cases} \dot{b}(t)-(\alpha-\delta)b(t)-\beta(d(t)-a(t))=0 \\ \dot{c}(t)-(\delta-\alpha)c(t)-\gamma(a(t)-d(t))=0 \end{cases} \tag{3.4}$$

$$b(0)=k, \qquad c(0)=0 \tag{3.5}$$

$$\begin{cases} \phi(t)=\dot{a}(t)-\beta c(t)+\gamma b(t) \\ \psi(t)=\dot{d}(t)-\gamma b(t)+\beta c(t). \end{cases} \tag{3.6}$$

In this case, the equation (1.1) *with the coefficient* (3.1) *can be solved explicitly.*

The another application of Corollary 1 to the stability theory of the homogeneous linear system of differential equations (1.1) will be published later on.

References

[1] H. A. Coppel: Dichotomies and reducibility. Journal of Diff. Eqs., **3**, 500–521 (1967).

[2] L. Markus and H. Yamabe: Global stability criteria for differential systems. Osaka Math. J., **12**, 305–317 (1960).

[3] M. Y. Wu: An extension of "A new method of computing the state transition matrix of linear time-varying systems." IEEE Trans. Automat. Contr., AC-**19**, 619–620 (1974).

[4] V. I. Zubov: Mathematical Methods for the Study of Automatic Control Systems. Pergamon Press (1962).