64. *On the flat conformal differential geometry I.*

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§ 0. *Introduction.*

The generalization of conformal geometry was first discussed by H. Weyl[1] who considered a conformal transformation of the form $\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu}$ of the fundamental tensor of a Riemannian space. The theorems which state properties invariant under this transformation of the fundamental tensor constitute the conformal geometry of Riemannian spaces. The conformal geometry of Riemannian spaces was studied, from this point of view, especially by geometers of the American School, A. Fialkow,[2] V. A. Hoyle,[3] J. Levine,[4] V. Modesitt,[5] J. M. Thomas,[6] T. Y. Thomas,[7] J. Vanderslice,[8] O. Veblen[9] and others. The relative tensor $G_{\mu\nu}=g_{\mu\nu}/g^{\frac{1}{n}}$, where g is the determinant formed with components $g_{\mu\nu}$ of the fundamental tensor, being invariant under a conformal transformation, T. Y. Thomas defined the conformal geometry as the theory of invariants of relative quadratic differential form $G_{\mu\nu}d\xi^\mu d\xi^\nu$.

The study of conformal geometry of Riemannian spaces was continued in

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(1) H. Weyl: Zur Infinitesimalgeometrie. Einordnung der projektiven und konformen Auffassung. Göttinger Nachrichten, (1921), 99–112.

(2) A. Fialkow: Conformal geodesics. Trans. Amer. Math. Soc., **45** (1939), 443–473; The conformal theory of curves. Proc. Nat. Acad. Sci. U.S.A., **26** (1940), 437–439.

(3) V. A. Hoyle: Some problems in conformal geometry. Diss. Princeton Univ., (1931).

(4) J. Levine: Conformal-affine connections. Proc. Nat. Acad. Sci. U.S.A., **21** (1935), 165–167; New identities in conformal geometry. Duke Math. Journ., **1** (1935), 173–184; Conformal scalars. Bull. Amer. Math. Soc., **42** (1936), 115–124; Groups of motions in conformally flat spaces. Ibid. 418–422.

(5) V. Modesitt: Some singular properties of conformal transformations between Riemannian spaces. Amer. Journ. Math., **60** (1938), 325–336.

(6) J. M. Thomas: Conformal correspondence of Riemannian spaces. Proc. Nat. Acad. Sci. U.S.A., **11** (1925), 257–259; Conformal invariants. Ibid. **12** (1926), 389–393.

(7) T. Y. Thomas: Invariants of relative quadratic differential forms. Ibid. **11** (1925), 722–725; On conformal geometry. Ibid. **12** (1926), 352–359; Conformal, tensors (First note). Ibid. **18** (1932), 103–112; Conformal tensors (Second note). Ibid. 188–193; The differential invariants of generalized spaces. Cambridge University Press, (1935).

(8) J. Vanderslice: Conformal tensor invariants. Proc. Nat. Acad. Sci. U.S.A., **20** (1934), 672–676.

(9) O. Veblen: Conformal tensors and connections. Ibid. **14** (1928), 735–745; Formalism for conformal geometry. Ibid. **21** (1935), 168–173.

our country by Y. Mutô,[10] S. Sasaki[11] and the present author[12]

V. Hlavatý[13] also showed that we can study the conformal geometry of Riemannian spaces in the light of Weyl's work which established a geometry in which the covariant derivative of the fundamental tensor is proportional to the fundamental tensor itself. The present author[14] gave its application and discussed conformal Frenet formulae.

On the other hand, E. Cartan[15] introduced first the notion of the connection which furnishes a powerful tool to give a really geometrical aspect to the theory of Riemannian spaces and to open a way permitting a generalization of geometries defined in Erlanger Programm in the direction suggested by the geometrical interpretation of T. Levi-Civita on covariant derivative. But unfortunately, the paper of E. Cartan was left isolated until the present author[16] studied the relationship between the theory of E. Cartan and that of the American School.

The study of the theory of spaces with conformal connection was performed, with the use of the method of repère mobile of E. Cartan, especially in our country by Y. Muto,[17] S. Sasaki[18] and the present author.[19] The present author[20] remarked that the conformal geometry of Riemannian spaces may also be studied from point of view of E. Cartan, if we introduce a normal conformal

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10) K. Yano and Y. Mutô: Sur le théorème fondamental dans la géométrie conforme des sous-espaces riemanniens. Proc. Physico-Math. Soc. Japan, **24** (1942), 437–449.

11) S. Sasaki: Some theorems on conformal transformations of Riemannian spaces. Ibid. **18** (1936), 572–578; Another demonstration of a theorem of J. A. Schouten on totally umbilical hypersurfaces in a Riemannian space. Tensor, **2** (1939), 25–29.

12) K. Yano: Sur les équations de Gauss dans la géométrie conforme des espaces de Riemann. Proc., **15** (1939), 247–252; Sur les équations de Codazzi dans la géométrie conforme des espaces de Riemann. Ibid. **15** (1939), 340–344; Conformally separable quadratic differential forms. Ibid. **16** (1940), 83–86; Sur quelques propriétés conformes de V_l dans V_m dans V_n. Ibid. 173–177; Projective parameters in projective and conformal geometries. Ibid. **20** (1944), 45–53.

13) V. Hlavatý: Zur Konformgeometrie, I, II, III. Proc. Akad. Amsterdam, **38** (1935), 281–286; 706–708; 1006–1011; Système de connexions de M. Weyl. Acad. Tchèque Sci. Bull. Int., **37** (1936), 181–184.

14) K. Yano: Sur la connexion de Weyl-Hlavatý et la géométrie conforme. Proc., **15** (1939), 116–120; Sur la connexion de Weyl-Hlavatý et ses applications à la géométrie conforme. Proc. Physico-Math. Soc. Jap., **22** (1940), 595–621.

15) E. Cartan: Les espaces à connexion conforme. Annales de la Soc. Polonaise de Math., **2** (1923), 171–221.

16) K. Yano: Sur la théorie des espaces à connexion conforme. Journal of the Faculty of Science, Imperial University of Tokyo, Vol. 4, Part, 1, (1939), 1–59.

17) Y. Mutô: On the generalized circles in the conformally connected manifold. Proc., **15** (1939), 23–26; On some properties of hypersurfaces in a conformally connected manifold. Proc. Physico-Math. Soc. Jap., **21** (1939), 615–625; On some properties of umbilical points of hypersurfaces. Proc., **22** (1940), 79–82; On some properties of subspaces in a conformally connected manifold. Proc. Physico-Math. Soc. Jap., **22** (1940), 621–638.

connection in a Riemannian space and if we use the repère mobile introduced for the first time by O. Veblen[21].

To study the generalized conformal geometry, we may use the projective method. J. A. Schouten and J. Haantjes[22] tried to construct the generalized conformal geometry with the use of D. van Dantzig's curvilinear homogeneous coordinates. Y. Mutô and the present author[23] tried also this study in the light of O. Veblen's work on the generalized projective geometry. But, it seems to the author that these studies are not yet complete to develop wholly the theory.

It may be very interesting to study the relationship between the theory of E. Cartan, that of the American School and that of the School of Delft. But it will be difficult enough to try this study for the generalized conformal geometry in the stage of development of the theory.

In the series of the following papers, we shall try this study for the flat conformal differential geometry. We shall begin with the study of the fundamental differential equations of flat conformal geometry.

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18) S. Sasaki: On the theory of curves in a curved conformal space. Sci. Rep. Tôhoku Imp. Univ., **27** (1939), 392-409; On the theory of surfaces in a curved conformal spaces. Ibid. **28** (1940), 261-285; Geometry of conformal connexion. Ibid. **28** (1940), 219-267; On a remarkable property of umbilical hypersurfaces in the geometry of the normal conformal connection. Ibid. **29** (1940), 412-422; On conformal normal coordinates. Ibid. **30** (1941), 71-80; On the spaces with normal conformal connexions whose groups of holonomy fix a point or a hypersphere, I, II, III. Jap. Journ. Math., **18** (1943), 615-622, 623-633, 791-798.

19) K. Yano: Remarques relatives à la théorie des espaces à connexion conforme. C.R., **206** (1938), 560-562; Sur les circonférences généralisées dans un espace à connexion conforme. Proc., **14** (1938), 329-332; Sur les équations fondamentales dans la géométrie conforme des sous-espaces. Ibid. **19** (1943), 326-334; Sur une application du tenseur conforme C_{jk} et du scalaire conforme C. Ibid. 335-340; Conformal and concircular geometries in Einstein spaces. Ibid. 444-453.

K. Yano and Y. Mutô: A projective treatment of a conformally connected manifold. Proc. Physico-Math. Soc. Jap., **21** (1939), 270-286; Sur la théorie des hypersurfaces dans un espace à connexion conforme. Proc., **16** (1940), 266-273; Sur la théorie des hypersurfaces dans un espace à connexion conforme. Jap. Journ. Math., **17** (1941), 229-288; Sur la théorie des espaces à connexion conforme normale et la géométrie conforme des espaces de Riemann. Proc., **17** (1941), 87-94; Sur la théorie des espaces à connexion conforme normale et la géométrie conforme des espaces de Riemann. Journal of the Faculty of Science, Imperial University of Tokyo, Vol. 4, Part, 3, (1941), 117-169; On the conformal arc length. Proc., **17** (1941), 318-322; On the generalized loxodromes in the conformally connected manifold. Ibid. 455-460; On the curves developable on two dimensional spheres in the conformally connected manifold. Ibid. **18** (1942), 222-226.

K. Yano and S. Sasaki: Sur les espaces à connexion conforms normale dont les groupes d'holonomie fixent une sphère à un nombre quelconque de dimensions, I. Ibid. **20** (4944), 525-535.

20) K. Yano and Y. Mutô: Sur la théorie des espaces à connexion conforme normale et la géométrie conforme des espaces de Riemann, loc. cit. See (19).

21) O. Veblen: Formalism for conformal geometry, loc. cit. See (9).

§ 1. *Fundamental differential equations of flat conformal geometry.*

1°. *Fundamental equations.*

Let us consider an n-dimensional conformal space C_n and suppose that a point A of C_n is represented by $n+2$ homogeneous coordinates $(a^0, a^1, \dots, a^n, a^\infty)$, not vanishing simultaneously, and satisfying the relation

$$(1.1) \qquad A\cdot A \equiv a^1a^1 + a^2a^2 + \dots + a^na^n - a^0a^\infty = 0.$$

If $A\cdot A \gtreqless 0$, A represents an $(n-1)$-dimensional sphere or hypersphere of the space. Consequently, if $A\cdot A=0$, A may be called point-hypersphere of the space. The inner product of two hyperspheres A and B being defined by

$$(1.2) \qquad A\cdot B = a^1b^1 + a^2b^2 + \dots + a^nb^n - a^0b^\infty - a^\infty b^0,$$

the angle θ under which two hyperspheres A and B cut mutually is given by

$$(1.3) \qquad \cos\theta = A\cdot B/\sqrt{A\cdot A}\cdot\sqrt{B\cdot B}.$$

The condition that A be a point is represented by $A\cdot A=0$, the condition that a point A be on a hypersphere B by $A\cdot A=0$, $A\cdot B=0$, the condition that two hyperspheres A and B be orthogonal by $A\cdot B=0$.

Moreover, A and αA represent the same hypersphere, and $\alpha A+\beta B$ represents a pencil of hyperspheres involving A and B, α and β being arbitrary real numbers.

Now, we shall introduce a curvilinear coordinates system $(\xi^1, \xi^2, \dots, \xi^n)$ and put

$$(1.4) \qquad \begin{cases} a^0 = \sigma f^0(\xi^1, \xi^2, \dots, \xi^n), \\ a^1 = \sigma f^1(\xi^1, \xi^2, \dots, \xi^n), \\ \dots\dots\dots\dots \\ a^n = \sigma f^n(\xi^1, \xi^2, \dots, \xi^n), \\ a^\infty = \sigma f^\infty(\xi^1, \xi^2, \dots, \xi^n), \end{cases}$$

where we suppose that a's satisfy the coodition (1.1) and the rank of the matrix

$$(1.5) \qquad \begin{pmatrix} a^0, & a^1, \dots, & a^n, & a^\infty \\ \dfrac{\partial a^0}{\partial \xi^1}, & \dfrac{\partial a^1}{\partial \xi^1}, \dots, & \dfrac{\partial a^n}{\partial \xi^1}, & \dfrac{\partial a^\infty}{\partial \xi^1} \\ \dots\dots\dots\dots \\ \dfrac{\partial a^0}{\partial \xi^n}, & \dfrac{\partial a^1}{\partial \xi^n}, \dots, & \dfrac{\partial a^n}{\partial \xi^n}, & \dfrac{\partial a^\infty}{\partial \xi^n} \end{pmatrix}$$

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(22) J. A. Schouten and J. Haantjes: Über allgemeine konforme Geometrie in Projektiven Behandlung, I. Proc. Akad. Amsterdam, **38** (1935), 706–708; II, Ibid. **39** (1936), 27; Beiträge zur allgemeinen (gekrümmten) konformen Differentialgeometrie, I. Math. Ann. **112** (1936), 594–629; II, Ibid. **113** (1936), 568–583.

(23) K. Yano and Y. Mutô: A projective treatment of a conformally connected manifold, loc. cit. See (19).

be $n+1$.

If we differentiate the relation $A\cdot A=0$ with respect to ξ^λ $(\alpha, \beta, \gamma, \dots, \lambda, \mu, \nu, \dots = 1, 2, \dots, n)$, we find

$$(1.6) \qquad A\cdot\frac{\partial A}{\partial\xi^\lambda}=0.$$

Then, we see that n hyperspheres

$$(1.7) \qquad A_\lambda=\frac{\partial A}{\partial\xi^\lambda}$$

thus defined all pass through the point-hypersphere A. If we put

$$(1.8) \qquad g_{\mu\nu}=A_\mu\cdot A_\nu,$$

$g_{\mu\nu}$ are functions of ξ^λ whose determinant is different from zero. The function σ in (1.4) being arbitrary, the functions $g_{\mu\nu}$ are only defined except a common factor.

The hyperspheres $A_1, A_2, \dots, A_n$ passing through all the same point $A=A_0$, they must pass through also another same point A_∞. If we choose suitably the factor of A_∞, we may put

$$(1.9) \qquad A_0\cdot A_\infty=-1, \quad A_\lambda\cdot A_\infty=0, \quad A_\infty\cdot A_\infty=0.$$

Now, it will be easily seen that the $n+2$ hyperspheres that we have defined are linearly independent, and any hypersphere P of the space C_n may be represented as a linear combination of $n+2$ hyperspheres A_0, A_λ, A_∞, thus

$$P=p^0A_0+p^\lambda A_\lambda+p^\infty A_\infty.$$

Consequently, the $n+2$ hyperspheres A_0, A_λ, A_∞ defined at every point of the space C_n may be considered as forming a moving $(n+2)$-spherical reference system or a repère mobile at a point A_0.

Let us differentiate $n+2$ hyperspheres A_0, A_μ, A_∞ with respect to ξ^ν, obtaining $n(n+2)$ hyperspheres

$$\frac{\partial A_0}{\partial\xi^\nu}, \quad \frac{\partial A_\mu}{\partial\xi^\nu} \quad \text{and} \quad \frac{\partial A_\infty}{\partial\xi^\nu}.$$

These hyperspheres must be represented as linear combinations of A_0, A_λ and A_∞. Thus we have

$$(1.10) \quad \begin{cases} \text{(i)} & \dfrac{\partial A_0}{\partial\xi^\nu}=\Pi^0_{0\nu}A_0+\Pi^\lambda_{0\nu}A_\lambda+\Pi^\infty_{0\nu}A_\infty, \\[2ex] \text{(ii)} & \dfrac{\partial A_\mu}{\partial\xi^\nu}=\Pi^0_{\mu\nu}A_0+\Pi^\lambda_{\mu\nu}A_\lambda+\Pi^\infty_{\mu\nu}A_\infty, \\[2ex] \text{(iii)} & \dfrac{\partial A_\infty}{\partial\xi^\nu}=\Pi^0_{\infty\nu}A_0+\Pi^\lambda_{\infty\nu}A_\lambda+\Pi^\infty_{\infty\nu}A_\infty. \end{cases}$$

From the equations (1.7) and (1.10) (i), we see that

$$(1.11) \qquad \Pi^0_{0\nu}=0, \quad \Pi^\lambda_{0\nu}=\delta^\lambda_\nu, \quad \Pi^\infty_{0\nu}=0.$$

Differentiating the relations $A_\mu \cdot A_\nu = g_{\mu\nu}$ and $A_0 \cdot A_\nu = 0$ with respect to ξ^ω, and substituting the equations (1.10) (i) and (ii) in the resulting equations, we find

$$\Pi^\lambda_{\mu\omega} g_{\lambda\nu} + \Pi^\lambda_{\nu\omega} g_{\mu\lambda} = \frac{\partial g_{\mu\nu}}{\partial \xi^\omega}$$

and

$$g_{\nu\omega} - \Pi^\infty_{\nu\omega} = 0,$$

respectively, from which we have

$$(1.12) \qquad \Pi^\lambda_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}, \qquad \Pi^\infty_{\mu\nu} = g_{\mu\nu},$$

$\Pi^\lambda_{\mu\nu}$ in (1.4) being symmetric with respect to μ and ν, where $\left\{ {\lambda \atop \mu\nu} \right\}$ denotes the Christoffel symbols.

Differentiating finally the relations

$$A_0 \cdot A_\infty = -1, \quad A_\mu \cdot A_\infty = 0, \quad A_\infty \cdot A_\infty = 0$$

with respect to ξ^ν, and substituting (1.10) (i), (ii) and (iii) in the resulting equations, we can easily obtain

$$(1.13) \qquad \Pi^0_{\infty\nu}=0, \; \Pi^\lambda_{\infty\nu} = g^{\lambda\mu}\Pi^0_{\mu\nu}, \; \Pi^\infty_{\infty\nu}=0.$$

Substituting (1.11), (1.12) and (1.13) in (1.10), we obtain finally

$$(1.14) \qquad \begin{cases} \dfrac{\partial A_0}{\partial_0 \xi^\mu} = A_\mu, \\[2ex] \dfrac{\partial A_\mu}{\partial \xi^\nu} = \Pi^0_{\mu\nu} A_0 + \left\{ {\lambda \atop \mu\nu} \right\} A_\lambda + g_{\mu\nu} A_\infty. \\[2ex] \dfrac{\partial A_\infty}{\partial \xi^\nu} = \Pi^\lambda_{\infty\nu} A_\lambda. \end{cases}$$

Thus, homogeneous coordinates of a point A_0 defined as functions of curvilinear coordinates ξ^λ must satisfy the above equations. New homogeneous coordinates obtained from old ones by a linear homogeneous transformation must satisfy also the same differential equations. Consequently we call (1.14) the fundamental equations of flat conformal geometry.

2°. *Transformation law of coefficients of the fundamental equations.*

The point A_0 being defined by homogeneous coordinates,

$$\bar{A}_0 = \phi A_0$$

represents the same point as A_0. By this change of factor, the hyperspheres A_μ are transformed into $\bar{A}_\mu$ following the formulae

$$\bar{A}_\mu = \phi(\phi_\mu A_0 + A_\mu),$$

where

$$\phi_{\mu}=\frac{\partial \log \phi}{\partial \xi^{\mu}}.$$

To obtain the transformation law of A_{∞}, we put

$$\overline{A}_{\infty}=\alpha^{0}A_{0}+\alpha^{\lambda}A_{\lambda}+\alpha^{\infty}A_{\infty},$$

then, from the conditions

$$\overline{A}_{0}\overline{A}_{\infty}=-1, \quad \overline{A}_{\lambda}\overline{A}_{\infty}=0, \quad \overline{A}_{\infty}\overline{A}_{\infty}=0,$$

we obtain

$$\alpha^{0}=\frac{1}{2\phi}g^{\mu\nu}\phi_{\mu}\phi_{\nu}, \quad \alpha^{\lambda}=\frac{1}{\phi}g^{\lambda\mu}\phi_{\mu}, \quad \alpha^{\infty}=\frac{1}{\phi},$$

consequently we have

$$\overline{A}_{\infty}=\frac{1}{\phi}\left(\frac{1}{2}g^{\mu\nu}\phi_{\mu}\phi_{\nu}A_{0}+g^{\mu\nu}\phi_{\mu}A_{\nu}+A_{\infty}\right).$$

Thus we have, for a change of factor,

$$(1.16)\quad \begin{cases} \overline{A}_{0}=\phi A_{0}, \\ \overline{A}_{\mu}=\phi(\phi_{\mu}A_{0}+A_{\mu}), \\ \overline{A}_{\infty}=\dfrac{1}{\phi}\left(\dfrac{1}{2}g^{\mu\nu}\phi_{\mu}\phi_{\nu}A_{0}+g^{\mu\nu}\phi_{\mu}A_{\nu}+A_{\infty}\right), \end{cases}$$

which give the transformation law of an $(n+2)$-spherical reference system $[A_{0}, A_{\lambda}, A_{\infty}]$ under a change of factor $\overline{A}_{0}=\phi A_{0}$.

We shall now consider the transformation law of the coefficients of the fundamental equations under this change of factor. Substituting (1.16) in the fundamental equations

$$\begin{cases} \dfrac{\partial \overline{A}_{0}}{\partial \xi^{\nu}}= \qquad \overline{A}_{\nu}, \\ \dfrac{\partial \overline{A}_{\mu}}{\partial \xi^{\nu}}=\overline{\Pi}^{0}_{\mu\nu}\overline{A}_{0}+\overline{\Pi}^{\lambda}_{\mu\nu}\overline{A}_{\lambda}+\overline{\Pi}^{\infty}_{\mu\nu}\overline{A}_{\infty}, \\ \dfrac{\partial \overline{A}_{\infty}}{\partial \xi^{\nu}}= \qquad \overline{\Pi}^{\lambda}_{\infty\nu}\overline{A}_{\lambda}, \end{cases}$$

we find, after some calculations,

$$(1.17)\quad \begin{cases} \Pi^{0}_{\mu\nu}=\Pi^{0}_{\mu\nu}+\dfrac{\partial \phi_{\mu}}{\partial \xi^{\nu}}-\phi_{\lambda}\Pi^{\lambda}_{\mu\nu}-\phi_{\mu}\phi_{\nu}+\dfrac{1}{2}g^{\alpha\beta}\phi_{\alpha}\phi_{\beta}g_{\mu\nu}, \\ \overline{\Pi}^{\lambda}_{\mu\nu}=\Pi^{\lambda}_{\mu\nu}+\delta^{\lambda}_{\mu}\phi_{\nu}+\delta^{\lambda}_{\nu}\phi_{\mu}-\phi^{\lambda}g_{\mu\nu}, \\ \overline{\Pi}^{\infty}_{\mu\nu}=\phi^{2}\Pi^{\infty}_{\mu\nu}, \\ \overline{\Pi}^{\lambda}_{\infty\nu}=\dfrac{1}{\phi^{2}}\left[\Pi^{\lambda}_{\infty\nu}+\dfrac{\partial \phi^{\lambda}}{\partial \xi^{\nu}}+\phi^{\mu}\Pi^{\lambda}_{\mu\nu}-\phi^{\lambda}\phi_{\nu}+\dfrac{1}{2}g^{\alpha\beta}\phi_{\alpha}\phi_{\beta}\delta^{\lambda}_{\nu}\right], \end{cases}$$

where $\phi^{\lambda}=g^{\lambda\mu}\phi_{\mu}$. These formulae give the transformation law of the coefficients of the fundamental equations.

Next, we shall consider transformation of curvilinear coordinates

$$(1.18) \qquad \bar{\xi}^\lambda = \bar{\xi}^\lambda(\xi^1, \xi^2, \ldots, \xi^n).$$

The analytical point A_0 being supposed to be invariant, it will be easily seen that the transformation of $[A_0, A_\lambda, A_\infty]$ under the change of coordinates (1.18) is given by

$$(1.19) \qquad \bar{A}_0 = A_0, \quad \bar{A}_\lambda = \frac{\partial \xi^\alpha}{\partial \bar{\xi}^\lambda} A_\alpha, \quad \bar{A}_\infty = A_\infty.$$

Substituting these equations in the fundamental equations written with barred quantities, we find, after some calculations.

$$(1.20) \qquad \begin{cases} \bar{\Pi}^0_{\mu\nu} = \dfrac{\partial \xi^\beta}{\partial \bar{\xi}^\mu} \dfrac{\partial \xi^\tau}{\partial \bar{\xi}^\nu} \Pi^0_{\beta\tau}, \\ \bar{\Pi}^\lambda_{\mu\nu} = \dfrac{\partial \bar{\xi}^\lambda}{\partial \xi^\alpha}\left(\dfrac{\partial \xi^\beta}{\partial \bar{\xi}^\mu} \dfrac{\partial \xi^\tau}{\partial \bar{\xi}^\nu} \Pi^\alpha_{\beta\tau} + \dfrac{\partial^2 \xi^\alpha}{\partial \bar{\xi}^\mu \partial \bar{\xi}^\nu} \right), \\ \bar{\Pi}^\infty_{\mu\nu} = \dfrac{\partial \xi^\beta}{\partial \bar{\xi}^\mu} \dfrac{\partial \xi^\tau}{\partial \bar{\xi}^\nu} \Pi^\infty_{\beta\tau}, \\ \bar{\Pi}^\lambda_{\infty\nu} = \dfrac{\partial \bar{\xi}^\lambda}{\partial \xi^\alpha} \dfrac{\partial \xi^\tau}{\partial \bar{\xi}^\nu} \Pi^\alpha_{\infty\tau}, \end{cases}$$

which give the transformation law of the coefficients of the fundamental equations under coordinates transformations. The equations show that, $\Pi^0_{\mu\nu}$, $\Pi^\infty_{\mu\nu}$ and $\Pi^\lambda_{\infty\nu}$ are components of tensors of second order, and $\Pi^\lambda_{\mu\nu}$ are components of an affine connection under a coordinate transformation.

3°. *Integrability conditions of the fundamental equations.*

We shall consider, in this Paragraph, the integrability conditions of the fundamental equations of flat conformal geometry.

From (1.15), we observe first that

$$(1.21) \qquad \Pi^0_{\mu\nu} = \Pi^0_{\nu\mu}, \quad \left\{ {\lambda \atop \mu\nu} \right\} = \left\{ {\lambda \atop \nu\mu} \right\}, \quad g_{\mu\nu} = g_{\nu\mu}.$$

Next, we differentiate the second equation of (1.14) with respect to ξ^ω and substitute (1.14) in the resulting equations. The right hand member of the equations thus obtained must be symmetric with respect to ν and ω, thus we have

$$\Omega^0_{\cdot\mu\nu\omega} = \frac{\partial \Pi^0_{\mu\nu}}{\partial \xi^\omega} - \frac{\partial \Pi^0_{\mu\omega}}{\partial \xi^\nu} + \left\{ {\alpha \atop \mu\nu} \right\} \Pi^0_{\alpha\omega} - \left\{ {\alpha \atop \mu\omega} \right\} \Pi^0_{\alpha\nu} = 0,$$

$$\Omega^\lambda_{\cdot\mu\nu\omega} = \frac{\partial \left\{ {\lambda \atop \mu\nu} \right\}}{\partial \xi^\omega} - \frac{\partial \left\{ {\lambda \atop \mu\omega} \right\}}{\partial \xi^\nu} + \left\{ {\alpha \atop \mu\nu} \right\} \left\{ {\lambda \atop \alpha\omega} \right\} - \left\{ {\alpha \atop \mu\omega} \right\} \left\{ {\lambda \atop \alpha\nu} \right\} + \Pi^0_{\mu\nu}\delta^\lambda_\omega - \Pi^0_{\mu\omega}\delta^\lambda_\nu$$
$$+ g_{\mu\nu}\Pi^\lambda_{\infty\omega} - g_{\mu\omega}\Pi^\lambda_{\infty\nu} = 0,$$

$$\Omega^\infty_{\cdot\mu\nu\omega} = \frac{\partial g_{\mu\nu}}{\partial \xi^\omega} - \frac{\partial g_{\mu\omega}}{\partial \xi^\nu} + \left\{ {\alpha \atop \mu\nu} \right\} g_{\alpha\omega} - \left\{ {\alpha \atop \mu\omega} \right\} g_{\alpha\nu} = 0$$

by virtue of the linear independence of A_0, A_λ and A_∞. The third equation is an identity and does not give any condition.

If we apply the same method as above to the third eqation of (1.15), we shall find

$$\Omega^{\lambda}_{\cdot\infty\nu\omega}=\frac{\partial\Pi^{\lambda}_{\infty\nu}}{\partial\xi^{\omega}}-\frac{\partial\Pi^{\lambda}_{\infty\omega}}{\partial\xi^{\nu}}+\Pi^{\alpha}_{\infty\nu}\{^{\lambda}_{\alpha\omega}\}-\Pi^{\alpha}_{\infty\omega}\{^{\lambda}_{\alpha\nu}\}=0.$$

But, if we remark that $\Pi^{\lambda}_{\infty\nu}=g^{\lambda\mu}\Pi^{0}_{\mu\nu}$, we can easily show that

$$\Omega^{\lambda}_{\cdot\infty\nu\omega}=g^{\lambda\mu}\Omega^{0}_{\cdot\mu\nu\omega},$$

then, the above condition is a consequence of the condition $\Omega^{0}_{\cdot\mu\nu\omega}=0$.

Thus, we obtain

$$(1.22)\qquad \Omega^{0}_{\cdot\mu\nu\omega}=\Pi^{0}_{\mu\nu;\omega}-\Pi^{0}_{\mu\omega;\nu}=0,$$

$$(1.23)\qquad \Omega^{\lambda}_{\cdot\mu\nu\omega}=R^{\lambda}_{\cdot\mu\nu\omega}+\Pi^{0}_{\mu\nu}\delta^{\lambda}_{\omega}-\Pi^{0}_{\mu\omega}\delta^{\lambda}_{\nu}+g_{\mu\nu}\Pi^{\lambda}_{\infty\omega}-g_{\mu\omega}\Pi^{\lambda}_{\infty\nu}=0,$$

as integrability conditions of the fundamental equations, where a semicolon denotes the covariant derivative with respect to $\{^{\lambda}_{\mu\nu}\}$ and $R^{\lambda}_{\cdot\mu\nu\omega}$ the curvature tensor formed with Christoffel symbols $\{^{\lambda}_{\mu\nu}\}$.

Contracting in (1.23) with respect to λ and ω, we find

$$(1.24)\qquad \Omega^{\lambda}_{\cdot\mu\nu\lambda}=R_{\mu\nu}+(n-2)\Pi^{0}_{\mu\nu}+g_{\mu\nu}g^{\beta\tau}\Pi^{0}_{\beta\tau}=0,$$

where $R_{\mu\nu}$ is Ricci tensor $R_{\mu\nu}=R^{\lambda}_{\cdot\mu\nu\lambda}$. Multiplying this equation by $g^{\mu\nu}$ and summing up, we obtain

$$g^{\mu\nu}\Omega^{\lambda}_{\cdot\mu\nu\lambda}=R+2(n-1)g^{\beta\tau}\Pi^{0}_{\beta\tau}=0,$$

from which

$$g^{\beta\tau}\Pi^{0}_{\beta\tau}=-\frac{R}{2(n-1)},$$

where R is the scalar curvature $R=g^{\mu\nu}R_{\mu\nu}$.

Substituting this equation in (1.24), we find

$$(n-2)\Pi^{0}_{\mu\nu}=-R_{\mu\nu}+\frac{g_{\mu\nu}R}{2(n-1)}.$$

Thus, we have, if $n>2$,

$$(1.25)\qquad \Pi^{0}_{\mu\nu}=-\frac{R_{\mu\nu}}{n-2}+\frac{g_{\mu\nu}R}{2(n-1)(n-2)},$$

and consequently

$$(1.26)\qquad \Pi^{\lambda}_{\infty\nu}=-\frac{R^{\lambda}_{\cdot\nu}}{n-2}+\frac{\delta^{\lambda}_{\nu}R}{2(n-1)(n-2)}.$$

Substituting (1.25) and (1.26) in the equations (1.22) and (1.23), we obtain

$$(1.27)\qquad \Omega^{0}_{\cdot\mu\nu\omega}=-\frac{R_{\mu\nu;\omega}}{n-2}+\frac{g_{\mu\nu}R_{;\omega}}{2(n-1)(n-2)}+\frac{R_{\mu\omega;\nu}}{n-2}$$

$$-\frac{g_{\mu\omega}R_{;\nu}}{2(n-1)(n-2)}=0,$$

and

$$(1.28)\quad \Omega^{\lambda}_{\cdot\mu\nu\omega}=R^{\lambda}_{\cdot\mu\nu\omega}-\frac{1}{n-2}\left(R_{\mu\nu}\delta^{\lambda}_{\omega}-R_{\mu\omega}\delta^{\lambda}_{\nu}+g_{\mu\nu}R^{\lambda}_{\cdot\omega}-g_{\mu\omega}R^{\lambda}_{\cdot\nu}\right)$$
$$+\frac{R}{(n-1)(n-2)}\left(g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu}\right)=0$$

respectively. The equation (1.28) shows that the Weyl curvature tensor formed with our $g_{\mu\nu}$ must vanish.

By covariant differentiation and contraction, we have, from (1.28),

$$\Omega^{\lambda}_{\cdot\mu\nu\omega;\lambda}=R^{\lambda}_{\cdot\mu\nu\omega;\lambda}-\frac{1}{n-2}(R_{\mu\nu;\omega}-R_{\mu\omega;\nu}+g_{\mu\nu}R^{\lambda}_{\cdot\omega;\lambda}-g_{\mu\omega}R^{\lambda}_{\cdot\nu;\lambda})$$
$$+\frac{1}{(n-1)(n-2)}(g_{\mu\nu}R_{;\omega}-g_{\mu\omega}R_{;\nu}).$$

Substituting, in the above equation, the relations

$$R^{\lambda}_{\cdot\mu\nu\omega;\lambda}+R_{\mu\omega;\nu}-R_{\mu\nu;\omega}=0 \quad\text{and}\quad 2R^{\lambda}_{\cdot\omega;\lambda}=R_{;\omega}$$

deducible from Bianchi's identity, we obtain

$$\Omega^{\lambda}_{\cdot\mu\nu\omega;\lambda}=(n-3)\left[\frac{R_{\mu\nu;\omega}}{n-2}-\frac{g_{\mu\nu}R_{;\omega}}{2(n-1)(n-2)}-\frac{R_{\mu\omega;\nu}}{n-2}+\frac{g_{\mu\omega}R_{;\nu}}{2(n-1)(n-2)}\right],$$

from which

$$(1.29)\qquad \Omega^{\lambda}_{\cdot\mu\nu\omega;\lambda}=-(n-3)\Omega^{0}_{\cdot\mu\nu\omega}.$$

Thus when $n>3$, the condition $\Omega^{0}_{\cdot\mu\nu\omega}=0$ is a consequence of $\Omega^{\lambda}_{\cdot\mu\nu\omega}=0$. When $n=3$, it is well known that the Weyl's conformal curvature vanishes identically. Thus we have the

Theorem: *In order that $\Pi^{0}_{\mu\nu}$, $\{^{\lambda}_{\mu\nu}\}$, $g_{\mu\nu}$ and $\Pi^{\lambda}_{\infty\nu}=g^{\lambda\mu}\Pi^{0}_{\mu\nu}$ be coefficients of fundamental differential equations of flat conformal geometry, it is necessary and sufficient that:*

when $n=3$,

$$\Omega^{0}_{\cdot\mu\nu\omega}=\Pi^{0}_{\mu\nu;\omega}-\Pi^{0}_{\mu\omega;\nu}=0,$$

when $n>3$,

$$\Omega^{\lambda}_{\cdot\mu\nu\omega}=R^{\lambda}_{\cdot\mu\nu\omega}-\frac{1}{n-2}(R_{\mu\nu}\delta^{\lambda}_{\omega}-R_{\mu\omega}\delta^{\lambda}_{\nu}+g_{\mu\nu}R^{\lambda}_{\cdot\omega}-g_{\mu\omega}R^{\lambda}_{\cdot\nu})$$
$$+\frac{R}{(n-1)(n-2)}(g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu})=0,$$

$\Pi^{0}_{\mu\nu}$ being necessarily given by

$$\Pi^{0}_{\mu\nu}=-\frac{R_{\mu\nu}}{n-2}+\frac{g_{\mu\nu}R}{2(n-1)(n-2)}.$$

These conditions being stated in tensor form, it is evident that they are invariant under change of coordinates. If we effect a change of factor, it is easily shown that the tensors $\Omega^{0}_{\cdot\mu\nu\omega}$ and $\Omega^{\lambda}_{\cdot\mu\nu\omega}$ are transformed into $\overline{\Omega}^{0}_{\cdot\mu\nu\omega}$ and $\overline{\Omega}^{\lambda}_{\cdot\mu\nu\omega}$ respectively by the formulae

$$\overline{\Omega}^{0}_{\cdot\mu\nu\omega}=\Omega^{0}_{\cdot\mu\nu\omega}-\phi_{\lambda}\Omega^{\lambda}_{\cdot\mu\nu\omega}, \quad \overline{\Omega}^{\lambda}_{\cdot\mu\nu\omega}=\Omega^{\lambda}_{\cdot\mu\nu\omega},$$

thus the above conditions are also invariant under change of factor.

It is observed that the so-called projectively flat space does not necessarily coincide with the classical projective space,[24] but here the so-called conformally flat space coincide with the classical conformal space.

In the next Chapter, we shall introduce a canonical form of the fundamental equations and discuss transformation laws of their coefficients and the integrability conditions.

24) K. Yano: On the flat projective differential geometry. Jap. Journ. Math. (in press).