1. Equilibrium Potentials and Energy Integrals.

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1. Introduction

Frostman's theory[1] on equilibrium potentials of order α has been recently extended by Kunugui[2] to generalized potentials. The purpose of this paper is to study the same problem from another point of view.

For preparation we state some definitions on generalized potentials and capacities. Denote by Ω the whole Euclidean space, by $\delta(E)$ the diameter of a bounded Borel set E, by r_{PQ} the length of a segment PQ, and by D_E^m the family of non-negative mass distributions of total mass m on a bounded Borel set E; especially when $m=1$, we denote it by D_E simply. Let $\Phi(t)$ be a strictly monotone decreasing and continuous function defined in the interval $(0,\infty)$ such that $\lim\limits_{t\to 0}\Phi(t)=+\infty$. Given any mass distribution μ on E, we call the Lebesgue-Stieltjes integrals

$$\int_E \Phi(r_{PQ})d\mu(Q) \quad \text{and} \quad \iint_E \Phi(r_{PQ})d\mu(Q)d\mu(P)$$

the Φ-potentials and the Φ-energy integrals respectively with respect to μ. Put

$$V_E^\Phi = \inf_{\mu\in D_E}\sup_{P\in\Omega}\int_E \Phi(r_{PQ})d\mu(Q) \quad \text{and} \quad W_E^\Phi = \inf_{\mu\in D_E}\iint_E \Phi(r_{PQ})d\mu(Q)d\mu(P),$$

then it is easily seen that

$$\Phi[\delta(E)]\leqq V_E^\Phi\leqq+\infty \quad \text{and} \quad \Phi[\delta(E)]\leqq W_E^\Phi\leqq+\infty.$$

We define the Φ-capacity $C^\Phi(E)$ of E as follows; if $V_E^\Phi<+\infty$, then $C^\Phi(E)=\Phi^{-1}[V_E^\Phi]$, and if $V_E^\Phi=+\infty$, then $C^\Phi(E)=0$, where Φ^{-1} denotes the inverse function of Φ. Hereafter we shall write for the sake of simplicity V_E, W_E, $C(E)$ for V_E^Φ, W_E^Φ, $C_E^\Phi(E)$.

1) O. Frostman: Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. Thèse. Lund. 1935.

2) K. Kunugui: Sur quelques points de la théorie du potentiel. (I), (II). Proc. Jap. Acad. vol. 21-23.

2. Equilibrium potentials.

Let Ω be the ordinary space and $\Phi(t)$ satisfy the following conditions[3]:

α) $\Phi(t)$ is strictly monotone decreasing and continuous in $(0, \infty)$ and $\lim_{t\to 0}\Phi(t) = +\infty$.

β) $t\Phi(t)$ is convex in $(0, \infty)$. (What is the same, $\Phi(t)$ is convex in $\frac{1}{t}$ in $(0, \infty)$).

γ) $\int_0 t^2\Phi(t)dt < +\infty$ and $\overline{\lim_{r\to 0}}\dfrac{\int_0^r t^2\Phi(t)dt}{r^3\Phi(r)} < +\infty$.

Then, from β), we see that $\Phi(r_{PQ})$ is subharmonic in $\Omega-\{Q\}$ when Q is fixed in Ω. Accordingly, for any non-negative mass distribution μ on a bounded closed set F, the potential with respect to μ is subharmonic in each component of $\Omega-F$.

First, let us consider the maximum principle: Let $\Phi(t)$ satisfy the conditions α) and β) and $f(P)$ be continuous and superharmonic in Ω. If the potential $u(P)$ with respect to a non-negative mass distribution μ whose kernel is a bounded closed set F is $\leqq f(P)$ in F, then $u(P)\leqq f(P)$ in Ω. In case $\Phi(t)=\frac{1}{t^\alpha}$, it has been obtained by Yosida[4]. His proof is very elementary and interesting. The general case may be treated by a slight modification. Let us state the proof briefly. For any $\varepsilon>0$, take a closed subset F' of F such that $u(P)$ is continuous in F' and $\mu(F-F')<\varepsilon$. Then $\int_{E\cdot s(P,\delta_\varepsilon)}\Phi(r_{PQ})d\mu(Q)<\varepsilon$ [5] in F', where $s(P,\delta_\varepsilon)$ denotes any sphere with center $P\in F'$ and radius δ_ε (a constant). Therefore, $\int_{F'\cdot s(P,\delta_\varepsilon)}\Phi(r_{PQ})d\mu(Q)<\varepsilon$ in F'. Accordingly, $\int_{F'}\Phi(r_{PQ})d\mu(Q)$ is continuous in F' [5]; after all it become continuous in Ω.[5]

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3) If Ω is the plane, the conditions β) and γ) for $\Phi(t)$ must be replaced by the following:

β') $\Phi(t)$ is convex in $\log\frac{1}{t}$ in $(0, \infty)$.

γ') $\int_0 t\Phi(t)dt<+\infty$ and $\overline{\lim_{r\to 0}}\dfrac{\int_0^r t\Phi(t)dt}{r^2\Phi(r)}<+\infty$.

4) Y. Yosida: Sur le principle du Maximum dan la théorie du potentiel. Proc. Imp. Acad. Vol. 17. pp. 476-478.

5) O. Frostman: loc. cit. p. 26.

Let P^* be a point where $\int_{F'}\Phi(r_{PQ})d\mu(Q)-f(P)$ attains its maximum. Then $P^*\in F'$. Using this fact, for any point $P\in\Omega-F$, we easily see that

$$u(P)-f(P)\leqq(\Phi(l)-\Phi[\delta(F)]\cdot\mu(F-F'),$$

where l is the distance between P and F. Thus we obtain $u(P)\leqq f(P)$ in $\Omega-F$.

Using the maximum principle, we obtain at once the following two theorems.

Theorem A: *Let $\Phi(t)$ satisfy the conditions α) and β) and F be a bounded closed set of positive Φ-capacity. Then there exists $\mu_0\in D_F$ such that the Φ-potential $u_0(P)$ with respect to μ_0 is constant and equal to its maximum in F except a possible set of Φ-capacity 0.*

Theorem B: *Let $\Phi(t)$ and F be the same as above and $f(P)$ be continuous and superharmonic in Ω. Then there exists $\mu_0\in D_F$ such that the Φ-potential $u_0(P)$ with respect to μ_0 is $f(P)+\gamma$ in F except a possible set of Φ-capacity 0 and always $\leqq f(P)+\gamma$ in Ω, where γ is a suitable constant.*

Remark: The mass distribution μ_0 in Theorem A is what minimizes[6] energy integrals

$$I(\mu)=\iint_F\Phi(r_{PQ})d\mu(Q)d\mu(P)$$

with $\mu\in D_F$; while, the distribution μ_0 in Theorem B is what minimizes Gauss variations[7]

$$G(\mu)=\iint_F\Phi(r_{PQ})d\mu(Q)d\mu(P)-2\int_F f(P)d\mu(P)$$

with $\mu\in D_F$.

Remark: The following fact is very important: W_E coincides with V_E for any bounded Borel set E. It is easily proved from Theorem A and the properties[8] of W and V.

3. Poincaré's condition.

Given a bounded Borel set E and its limiting point P_0, we say that P_0 satisfies Poincaré's condition with respect to E, if there exists a cone such that its vertix is P_0 and its interior is contained

6) O. Frostman: loc. cit. p. 56.

7) S. Kametani: Positive definite integral quadratic forms and generalized potentials. Proc. Imp. Acad. vol. 20, p. 11.

8) O. Frostman loc. cit. pp. 49–52.

in E. Now, we shall replace Poincaré's condition by another potential-theoretic condition.

We obtain:

Theorem: *Let $\Phi(t)$ satisfy the conditions α) and γ), and let E any bounded Borel set, and P_0 its limiting point. If $\overline{\lim\limits_{s\to P_0}} \frac{m(Es)}{m(s)}>0$, then $\overline{\lim\limits_{s\to P_0}} \frac{V_s}{V_{Es}}>0$, where s denotes a sequence of closed spheres with center P_0.*

Proof: There exist δ_0 and m such that $\Phi(t)>0$ and $\frac{\int_0^r t^2\Phi(t)dt}{r^3\Phi(r)}<m$ in $(0, \delta_0)$. For any sphere s with center P_0, it is easily seen that $\int_s \Phi(r_{PQ})d\tau_Q$ is continuous in Ω and attains its maximum at P_0, where $d\tau_Q$ denotes a volume element at Q. Take a small sphere s with center P_0 and radius $r<\frac{1}{6}\delta_0$. Then

$$V_s\geqq W_s = \inf_{\mu\in D_s} \iint_s \Phi(r_{PQ})d\mu(Q)d\mu(P)\geqq \Phi(2r)$$
$$= 4\pi\cdot\Phi(2r)\cdot(2r)^3\cdot\frac{1}{24}\cdot\frac{1}{m(s)}$$

and

$$V_{Es} = \inf_{\mu\in D_{Es}} \sup_{P\in\Omega} \int_{Es} \Phi(r_{PQ})d\mu(Q)\leqq \frac{1}{m(Es)} \sup_{P\in\Omega} \int_{Es} \Phi(r_{PQ})d\tau_Q.$$

Let P^* be a point where $\int_{Es} \Phi(r_{PQ})d\tau_Q$, being continuous in Ω, attains its maximum. Then we easily see that $r_{P_0P^*}\leqq 3r$.
Hence

$$\int_{Es} \Phi(r_{P^*Q})\, d\tau_Q\leqq \int_s \Phi(r_{P^*Q})\, d\tau_Q\leqq \int_s \Phi(r_{P_0Q})\, d\tau_Q$$
$$= 4\pi\int_0^r t^2\Phi(t)\, dt<4\pi\int_0^{2r} t^2\Phi(t)dt.$$

Therefore

$$V_{Es}<\frac{4\pi}{m(Es)}\int_0^{2r} t^2\Phi(t)dt.$$

Accordingly

$$\frac{V_s}{V_{Es}}>\frac{1}{24}\cdot\frac{(2r)^3\cdot\Phi(2r)}{\int_0^{2r} t^2\Phi(t)dt}\cdot\frac{m(Es)}{m(s)}>\frac{1}{24m}\cdot\frac{m(Es)}{m(s)}\cdot$$

Thus we obtain

$$\overline{\lim_{s\to P_0}}\frac{V_s}{V_{Es}}\geqq\frac{1}{24m}\overline{\lim_{s\to P_0}}\frac{m(Es)}{m(s)}>0.$$

Hereafter we shall say that P_0 satisfies generalized Φ-Poincaré's condition with respect to E if $\overline{\lim_{s\to P_0}}\frac{V_s}{V_{Es}}>0.$

Theorem: *Let $\Phi(t)$ be the same as above, $u(P)$ a potential with respect to a non-negative mass distribution μ on a bounded closed set F, E any bounded Borel set, and P_0 its limiting point. If P_0 satisfies generalized Φ-Poincaré's condition with respect to E, then holds*

$$u(P_0)=\overline{\lim_{E\ni P\to P_0}}u(P).$$

Proof: For any $\lambda\leqq 1$ and $r\leqq\delta_0$, we obtain $\Phi(\lambda r)\leqq\frac{3m}{\lambda^3}\Phi(r)$, since

$$m\cdot r^3\Phi(r)>\int_0^r t^2\Phi(t)\,dt\geqq\int_0^{\lambda r}t^2\Phi(t)dt>\Phi(\lambda r)\cdot\int_0^{\lambda r}t^2dt=\frac{\lambda^3r^3}{3}\Phi(\lambda r).$$

Therefore, for any $\varepsilon<\mathrm{Min}(\Phi^{-1}(m),1)$, we get

$$\Phi(\varepsilon r)<\frac{3m}{\varepsilon^3}\Phi(r)<\frac{3}{\varepsilon^3}\Phi(\varepsilon)\cdot\Phi(r).$$

We have only to prove our theorem in the case when $P_0\in F_0$ and $u(P_0)<+\infty$, where F_0 denotes the kernel of μ. We can take a sphere S_0 with center P_0 and radius $R_0<\delta_0$ such that

$$\int_{S_0}\Phi(r_{P_0Q})d\mu(Q)<\varepsilon^6.$$

Put $\quad u_1(P)=\int_{S_0}\Phi(r_{PQ})d\mu(Q)$ and $u_2(P)=\int_{F-S_0}\Phi(r_{PQ})d\mu(Q)$.

Take a concentric sphere $s_0\subset S_0$ such that $u_2(P)<u_2(P_0)+\varepsilon$ in s_0. Let $\overline{\lim_{s\to P_0}}\frac{V_s}{V_{Es}}>K>0$. Then there exists a sequence of concentric spheres $\{s_n\}$ such that $s_0\supset s_1\supset s_2\cdots\to P_0$ and $\frac{V_{s_n}}{V_{Es_n}}>K>0$.

Take any s_n with radius $r_n<\varepsilon R_0$ and its concentric sphere S_n with radius $R_n\left(=\frac{r_n}{\varepsilon}\right)$, then $s_n\subset S_n\subset S_0$ and $\frac{\Phi(r_n)}{\Phi(R_n)}<\frac{3}{\varepsilon^3}\Phi(\varepsilon)$.

As $C(Es_n)>0$, there exists $\mu_n\in D_{Es_n}$ such that

$$\sup_{P\in\Omega}\int_{Es_n}\Phi(r_{PQ})d\mu_n(Q)<2V_{Es_n}.$$

$$\int_{Es_n} u(P)\, d\mu_n(P) = \int_{Es_n} u_1(P) d\mu_n(P) + \int_{Es_n} u_2(P) d\mu_n(P)$$

$$\int_{Es_n} u_2(P) d\mu_n(P) < u_2(P_0) + \varepsilon < u(P_0) + \varepsilon$$

$$\int_{Es_n} u_1(P) d\mu_n(P) = \int_{S_0} d\mu(Q) \int_{Es_n} \Phi(r_{PQ})\, d\mu_n(P)$$

$$= \int_{S_0 - S_n} d\mu(Q) \int_{Es_n} \Phi(r_{PQ})\, d\mu_n(P) + \int_{S_n} d\mu(Q) \int_{Es_n} \Phi(r_{PQ})\, d\mu_n(P).$$

We shall now evaluate the last two terms of the above expression. The second term is

$$< 2 V_{Es_n} \cdot \mu(S_n) < \frac{2}{K} V_{S_n} \cdot \mu(S_n)$$

$$< \frac{2}{K} \cdot \frac{\varepsilon^6}{\Phi(R_n)} \cdot \sup_{P \in \Omega} \frac{1}{\frac{4}{3}\pi r_n^3} \int_{S_n} \Phi(r_{PQ}) d\tau_Q$$

$$= \frac{6}{K} \cdot \frac{\varepsilon^6}{\Phi(R_n)} \cdot \frac{1}{\frac{4}{3}\pi r_n^3} \int_{S_n} \Phi(r_{P_0Q}) d\tau_Q$$

$$= \frac{1}{K} \cdot \frac{\varepsilon^6}{\Phi(R_n)} \cdot \frac{\int_0^{r_n} t^2 \Phi(t) dt}{r_n^3} < \frac{6}{K} \cdot \frac{\varepsilon^6}{\Phi(R_n)} \cdot m \Phi(r_n)$$

$$< \frac{6m}{K} \varepsilon^6 \cdot \frac{3}{\varepsilon^3} \Phi(\varepsilon) = \frac{18m}{K} \varepsilon^3 \Phi(\varepsilon).$$

Next, for any $P \in s_n$ and any $Q \in S_0 - S_n$,

$$\frac{r_{P_0Q}}{r_{PQ}} = 1 + \frac{r_{P_0Q} - r_{PQ}}{r_{PQ}} \leqq 1 + \frac{r_{P_0P}}{r_{P_0Q} - r_{P_0P}} \leqq 1 + \frac{r_n}{R_n - r_n}$$

$$= \frac{R_n}{R_n - r_n} = \frac{1}{1 - \varepsilon}.$$

Hence,

$$r_{PQ} \geqq (1-\varepsilon) r_{P_0Q}, \quad \Phi(r_{PQ}) \leqq \Phi[(1-\varepsilon) r_{P_0Q}] \leqq \frac{3m}{(1-\varepsilon)^3} \Phi(r_{PQ}),$$

and

$$\int_{Es_n} \Phi(r_{PQ}) d\mu_n(P) \leqq \frac{3m}{(1-\varepsilon)^3} \Phi(r_{P_0Q}).$$

Therefore, the first term of the above expression is

$$\leqq \frac{3m}{(1-\varepsilon)^3} \int_{S_0 - S_n} \Phi(r_{P_0Q})\, d\mu(Q) < \frac{3m}{(1-\varepsilon)^3} \int_{S_0} \Phi(r_{P_0Q})\, d\mu(Q) < \frac{3m}{(1-\varepsilon)^3} \cdot \varepsilon^6.$$

Thus

$$\int_{Es_n} u(P)d\mu_n(P) < u(P_0) + A(\varepsilon),$$

where

$$A(\varepsilon) = \varepsilon + \frac{3m}{(1-\varepsilon)^3}\,\varepsilon^6 + \frac{18m}{K}\,\varepsilon^3 \Phi(\varepsilon).$$

Therefore, there exists a sequence of points $\{P_n\}$ such that $E \ni P_n \to P_0$ and $u(P_n) < u(P_0) + A(\varepsilon)$. Accordingly,

$$\varliminf_{E \ni P \to P_0} u(P) \leqq u(P_0) + A(\varepsilon).$$

Making $\varepsilon \to 0$, we obtain $\varliminf_{E \ni P \to P_0} u(P) \leqq u(P_0)$.

We have $\varliminf_{E \ni P \to P_0} u(P) = u(P_0)$ from the lower semi-continuity of $u(P)$.

Theorem: *Let $\Phi(t)$ satisfy the conditions α), β), and γ). Then the equilibrium potential $u_0(P)$ in a bounded closed set F of positive Φ-capacity attains its maximum V_F at any point, of F, which satisfies generalized Φ-Poincaré's condition with respect to F.*

Proof: Put

$$F' = \{P;\, P \in F,\, u_0(P) = V_F\}.$$

Then $\qquad C(F - F') = 0.$

If $\varlimsup_{s \to P^*} \dfrac{V_s}{V_{Fs}} > 0$, then P^* is a limiting point of F' and $\varlimsup_{s \to P^*} \dfrac{V_s}{V_{F's}} > 0$.

Accordingly, $\qquad u_0(P^*) = \varliminf_{F' \ni P \to P^*} u(P) = V_F.$

Remark: Frostman has defined[9] the capacity density of E at P_0 by $\lim_{s \to P_0} \dfrac{C^\alpha(Es)}{C^\alpha(s)}$ in his theory of potentials of order α. For a sufficiently small sphere s,

$$V_{Es} = \Phi[C(Es)] = \Phi\left[\frac{C(Es)}{C(s)} \cdot C(s)\right] \leqq \frac{3m}{\left[\dfrac{C(Es)}{C(s)}\right]^3}\,\Phi[C(s)] = \frac{3m}{\left[\dfrac{C(Es)}{C(s)}\right]^3} \cdot V_s.$$

Hence we obtain

$$\varlimsup_{s \to P_0} \frac{V_s}{V_{Es}} \geqq \frac{1}{3m}\left[\varlimsup_{s \to P_0} \frac{C(Es)}{C(s)}\right]^3.$$

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9) O. Frostman: loc. cit. p. 57.

Thus our generalized Φ-Poincaré's condition contains the density of Φ-capacity by Frostman's definition.

4. Energy integrals.

Kunugui[10] has obtained the following important theorem: Let $\Phi(t)$ be a monotone increasing and convex function in $(0, \infty)$ and $\lim_{t\to 0} \Phi(t) = 0$. Then for any completely additive function σ of Borel sets on a bounded closed set F in the ordinary space, an energy integral

$$I(\sigma) = \iint_F \Phi\left(\frac{1}{r_{PQ}}\right) d\sigma(Q) d\sigma(P)$$

with respect to σ, if it exists, is always $\geqq 0$; especially the equality holds if and only if $\sigma \equiv 0$. In his proof he has used Fourier transformation of $\Phi\left(\frac{1}{r_{PQ}}\right)$. Here, we will give another proof by using Theorem A and B. We suppose that $\Phi(t)$ satisfies the conditions α) and β). For our proof we shall use the following six lemmas.

Lemma 1: *Let F_1 and F_2 be two disjoint bounded closed sets of positive Φ-capacity, μ and ν be two non-negative mass distributions of total mass unity whose kernels are F_1 and F_2 respectively. Then it is impossible that*

$$\gamma_1 = V_{F_1} - \int_{F_2} d\nu(P) \int_{F_1} \Phi(r_{PQ}) d\mu_0(Q)$$

and

$$\gamma_2 = V_{F_2} - \int_{F_1} d\mu(P) \int_{F_2} \Phi(r_{PQ}) d\nu_0(Q)$$

vanish simultaneously, where μ_0 and ν_0 denote equilibrium distributions on F_1 and F_2 respectively.

Proof: Evidently $\gamma_1 \geqq 0$ and $\gamma_2 \geqq 0$. We easily see that the equilibrium potential $u_0(P)$ on a bounded closed set F' of positive Φ-capacity is $< V_F$ in Ω_F^∞, where Ω_F^∞ denotes the component of Ω-F which contains the infinity. Either $F_1 \cdot \Omega_{F_2}^\infty$ or $F_2 \cdot \Omega_{F_1}^\infty$ is not empty. Suppose that $F_1 \cdot \Omega_{F_2}^\infty$ is not empty and P_0 is its arbitrary point. Then $V_{F_2} > \int_{F_2} \Phi(r_{P_0 Q}) d\nu_0(Q)$. Consequently, $V_{F_2} > \int_{F_2} \Phi(r_{PQ}) d\nu_0(Q)$ in some neighbourhood $U(P_0)$ of P_0.

10) K. Kunugui: loc. cit. (II).

Being

$$\mu[U(P_0)]>0, \quad V_{F_2}\cdot\mu\,[U(P_0)]>\int_{U(P_0)}d\mu(p)\int_{F_2}\Phi(r_{PQ})d\nu_0(Q).$$

But certainly

$$V_{F2}\cdot[\mu(F_1)-\mu\,[U(P_0)]]\geqq\int_{F_1-U(P_0)}d\mu\,(P)\int_{F_2}\Phi(r_{PQ})d\nu_0(Q).$$

Thus,

$$V_{F_2}>\int_{F_1}d\mu(P)\int_{F_2}\Phi(r_{PQ})d\nu_0(Q)$$

Lemma 2: *Let E be any bounded Borel sets and $\mu,\nu\in D_E$ such that*

$$\iint_E\Phi(r_{PQ})d\mu(Q)d\mu(P)<+\infty$$

and

$$\iint_E\Phi(r_{PQ})d\nu(Q)d\nu(P)<+\infty.$$

If

$$\int_E\Phi(r_{PQ})d\mu(Q)-\int_E\Phi(r_{PQ})d\nu(Q)$$

$\cong\delta$[11] *(a constant) in $\bar{E}$, then $\mu\equiv\nu$.*

Proof: Let $\sigma=\mu-\nu$ and suppose $\sigma\not\equiv0$. Let $\sigma=\mu'-\nu'$ be Hahn's decomposition of σ. Then there exists a closed sphere s[12] such that $\sigma(s)=a>0$, and we can take its concentric closed sphere S such that $S\supset s$ and $\nu'(S-s)<\varepsilon<a$. Let $f_1(P)=\int_{s^b}\frac{1}{r_{PM}}d\sigma_M$, s^b denoting a surface of s and $d\sigma_M$ a surface element at $M\in s^b$. Then, $f_1(P)\equiv A$ in s and $\equiv B$ on S^b, where both A and B are constants and $A>B>0$. $f_1(P)$ is continuous and superharmonic in Ω. Let $f_2(P)\equiv B$ in S and $\equiv f_1(P)$ in $\Omega-S$. Then $f_2(P)$ is also continuous and superharmonic in Ω. We see $C(E)>0$ from

$$\iint_E\Phi(r_{PQ})d\mu(Q)d\mu(P)<\infty$$

and

$$\iint_E\Phi(r_{PQ})d\nu(Q)d\nu(P)<+\infty.$$

By Theorem B, there exist $\mu_1,\mu_2\in D_{\bar{E}}$ such that

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11) $\cong$ means the coincidence except a possible set of Φ-capacity 0.

12) O. Frostman: loc. cit. p. 32.

$$\int_{\bar{E}} \Phi(r_{PQ})d\mu_1(Q) \geqq f_1(P)+\gamma_1$$

and

$$\int_{\bar{E}} \Phi(r_{PQ})d\mu_2(Q) \geqq f_2(P)+\gamma_2$$

in $\bar{E}$, where γ_1 and γ_2 are suitable constants. Put

$$f(P)=\int_{\bar{E}} \Phi(r_{PQ})\,d\mu_1(Q)-\int_{\bar{E}} \Phi(r_{PQ})d\mu_2(Q).$$

Then $f(P) \geqq f_1(P)-f_2(P)+\gamma$ in $\bar{E}$, where $\gamma=\gamma_1-\gamma_2$. Clearly, $f(P) \geqq A-B+\gamma$ in $\bar{E}\cdot s$, $\gamma \leqq f(P) \leqq A-B+\gamma$ in $\bar{E}\cdot(S-s)$ except a possible set of Φ-capacity 0, and $\geqq \gamma$ in $\bar{E}-S$. Next consider

$$\int_{E} f(P)d\sigma(P)=\int_{\bar{E}} f(P)d\sigma(P)=\int_{\bar{E}\cdot s}+\int_{\bar{E}\cdot(S-s)}+\int_{\bar{E}-S} f(P)d\sigma(P).$$

Of course σ cannot have any mass on a set of Φ-capacity 0, since

$$\iint_{E} \Phi(_{PQ})d\mu(Q)d\mu(P)<+\infty$$

and

$$\iint_{E} \Phi(r_{PQ})d\nu(Q)d\nu(P)<+\infty.$$

$$\int_{\bar{E}\cdot s} f(P)\,d\sigma(P)=(A-B+\gamma)\cdot\sigma(s),\quad \int_{\bar{E}-S} f(P)d\sigma(P)=\gamma\cdot\sigma(\bar{E}-S),$$

$$\int_{\bar{E}\cdot(S-s)} f(P)d\sigma(P)=\int_{\bar{E}\cdot(S-s)} f(P)\,d\mu'(P)-\int_{\bar{E}\cdot(S-s)} f(P)d\nu'(P)$$
$$\geqq \gamma\cdot\mu'(S-s)-(A-B+\gamma)\cdot\nu'(S-s)>-(A-B)\cdot\varepsilon+\gamma\cdot\sigma(S-s)$$

Therefore,

$$\int_{E} f(P)d\sigma(P)>(A-B)(a-\varepsilon)+\gamma\cdot\sigma(E)>\gamma\cdot\sigma(E)=0.$$

We have however, denoting $\kappa=\mu_1-\mu_2$,

$$\int_{E} f(P)\,d\sigma(P)=\int_{E} d\sigma(P)\int_{\bar{E}} \Phi(r_{PQ})d\kappa(Q)$$
$$=\int_{\bar{E}} d\kappa(Q)\int_{E} \Phi(r_{PQ})d\sigma(P)=0,$$

which is a contradiction.

Corollary: *The equilibrium distribution on a bounded closed set of positive Φ-capacity is unique.*

Hereafter, we suppose $\Phi(t)\geqq 0$ in $(0, \infty)$. When E_1 and E_2 are disjoint bounded Borel sets of positive Φ-capacity, for $\mu\in D_{E_1}$ and $\nu\in D_{E_2}$ we put

$$G[\mu\in D_{E_1}, \nu\in D_{E_2}]=\frac{\iint_{E_1}\Phi(r_{PQ})\,d\mu(Q)d\mu(P)\times\iint_{E_2}\Phi(r_{PQ})d\nu(Q)d\nu(P)}{\left[\int_{E_1}d\mu(P)\int_{E_2}\Phi(r_{PQ})d\nu(Q)\right]^2}.$$

So far as there is no confusion, we shall write $G(\mu, \nu)$ for

$$G[\mu\in D_{E_1}, \nu\in D_{E_2}].$$

Lemma 3: *Let E_1 and E_2 be two sets stated above. If a pair (μ^*, ν^*), $\mu^*\in D_{E_1}$ and $\nu^*\in D_{E_2}$, minimizes $G(\mu, \nu)$, then (μ^*, ν^*) has the following properties;*

i) $C\{P;\, P\in E_1, g_1(P)<0\}=0$ *and* $\mu^*\{P;\, P\in E_1, g_1(P)>0\}=0$,

ii) $C\{P;\, P\in E_2, g_2(P)<0\}=0$ *and* $\nu^*\{P;\, P\in E_2, g_2(P)>0\}=0$,

where

$$x=\iint_{E_1}\Phi(r_{PQ})\,d\mu^*(Q)d\mu^*(Q),$$

$$y=\iint_{E_2}\Phi(r_{PQ})d\nu^*(Q)d\nu^*(P),$$

$$z=\int_{E_1}d\mu^*(P)\int_{E_2}\Phi(r_{PQ})d\nu^*(Q),$$

$$g_1(P)=z^2\int_{E_1}\Phi(r_{PQ})d\mu^*(Q)-zx\int_{E_2}\Phi(r_{PQ})d\nu^*(Q)$$

and

$$g_2(P)=z^2\int_{E_2}\Phi(r_{PQ})d\nu^*(Q)-zy\int_{E_1}\Phi(r_{PQ})d\mu^*(Q)$$

under the assumption $0<x, y, z<+\infty$.

Proof: For any $\delta>0$, put

$$A=\{P;\, P\in E_1, g_1(P)>-\delta\} \text{ and } B=\{P;\, P\in E_1, g_1(P)<-2\delta\}.$$

Then $A\cdot B=0$ and $\mu^*(A)>0$ from $\int_{E_1}g_1(P)d\mu^*(P)=0$. Suppose $C(B)>0$. Then we can distribute a new mass τ on E_1 such that $\tau=-\mu^*$ on A, $\tau\geqq 0$ on B and $\tau(B)=\mu^*(A)=a>0$, $\tau\equiv 0$ on $E_1-(A+B)$ and $\sup_{P\in\Omega}\int_B\Phi(r_{PQ})d\tau(Q)<+\infty$. As $\mu^*+\varepsilon\tau\in D_{E_1}$ for any positive number $\varepsilon<1$, $G(\mu^*+\varepsilon\tau, \nu^*)\geqq G(\mu^*, \nu^*)$. Hence,

$$0 \leqq 2\varepsilon \int_{E_1} g_1(P)\,d\tau(P) + \varepsilon^2 \Big[z^2 \iint_{E_1} \Phi(r_{PQ})\,d\tau(Q)\,d\tau(P)$$
$$- x \Big(\int_{E_1} d\tau(P) \int_{E_2} \Phi(r_{PQ})\,d\nu^*(Q) \Big)^2 \Big]$$
$$\leqq -2\delta\alpha\varepsilon + \varepsilon^2 [\quad ,, \quad].$$

We can easily see that the coefficient of ε^2 is finite, and so the last side becomes <0 for some positive number $\varepsilon<1$, which is a conadiction. Therefore, $C(B)=0$. Making $\delta\to 0$, we obtain

$$C\{P;\, P\in E_1,\, g_1(P)<0\} = 0.$$

Accordingly, $\mu^*\{P;\, P\in E_1,\, g_1(P)<0\} = 0,$

since $\iint_{E_1} \Phi(r_{PQ})\,d\mu^*(Q)\,d\mu^*(P) < +\infty.$

Thus, $\mu^*\{P;\, P\in E_1,\, g_1(P)>0\} = 0.$

Similarly, we obtain

$$C\{P;\, P\in E_2,\, g_2(P)<0\} = 0$$

and $\nu^*\{P;\, P\in E_2,\, g_2(P)>0\} = 0.$

Lemma 4: *Let F_1 and F_2 be two disjoint closed sets of positive Φ-capacity, then $G(\mu,\nu)>1$ for any $\mu\in D_{F_1}$ and $\nu\in D_{F_2}$.*

Proof: Let $G^* = \inf_{\mu\in D_{F_1}\,\nu\in D_{F_2}} G(\mu,\nu)$. Then there exist $\{\mu_i\}\in D_{F_1}$ and $\{\nu_i\}\in D_{F_2}$ such that $G(\mu_i,\nu_i)\downarrow G^*$. We may suppose that both $\{\mu_i\}$ and $\{\nu_i\}$ are convergent. Let $\mu^*\in D_{F_1}$ and $\nu^*\in D_{F_2}$ be respectively their limiting mass distributions. Then $G^*\leqq G(\mu^*,\nu^*)$. On the other hand,

$$\iint_{F_1} \Phi(r_{PQ})\,d\mu^*(Q)\,d\mu^*(P) \leqq \varliminf_{i\to\infty} \iint_{F_1} \Phi(r_{PQ})\,d\mu_i(Q)\,d\mu_i(P),$$

$$\iint_{F_2} \Phi(r_{PQ})\,d\nu^*(Q)\,d\nu^*(P) \leqq \varliminf_{i\to\infty} \iint_{F_2} \Phi(r_{PQ})\,d\nu_i(Q)\,d\nu_i(P)$$

and

$$\int_{F_1} d\mu^*(P) \int_{F_2} \Phi(r_{PQ})\,d\nu^*(Q) = \lim_{i\to\infty} \int_{F_1} d\mu_i(P) \int_{F_2} \Phi(r_{PQ})\,d\nu_i(Q),$$

because $\Phi(r_{PQ})$ is bounded and continuous for $P\in F_1$ and $Q\in F_2$.

Hence, $G(\mu^*,\nu^*) \leqq \varliminf_{i\to\infty} G(\mu_i,\nu_i) = G^*.$

Therefore, $G^* = G(\mu^*,\nu^*)$. For μ^* and ν^*, let $x, y, z, g_1(P)$ and

$g_2(P)$ be as was stated in the previous lemma. Then evidently $0<x, y, z<+\infty$. Let

$$F'_1 = \{P;\, P\in F_1,\, g_1(P)\leqq 0\}$$

and F_1^* be the kernel of μ^*. We show that $F_1^*\subset F'_1$. Suppose that there exists $P_0\in F_1^*$ such that $g_1(P_0)>0$. Then $g_1(P)>0$ in some neighbourhood $U(P_0)$ of P_0 by the lower semi-continuity of $g_1(P)$ at P_0. As $\mu^*[U(P_0)]>0$, we obtain

$$\mu^*\{P;\, P\in F_1,\, g_1(P)>0\}>0.$$

which is a contradiction. Now, let μ_0^* be the equilibrium distribution on F_1^* (of positive Φ-capacity). Then

$$0\geqq\int_{F_1^*} g_1(P)d\mu_0^*(P) = z^2 V_{F_1^*}-zx\int_{F_2^*} d\nu^*(Q)\int_{F_1^*}\Phi(_{PQ})d\mu_0^*(P).$$

Hence we see $z^2\leqq zx$. Similarly, let ν_0^* be the equilibrium distribution on the kernel F_2^* of ν^*, then

$$0\geqq\int_{F_2^*} g_2(P)d\nu_0^*(P) = z^2 V_{F_2}^*-zy\int_{F_1^*} d\mu^*(Q)\int_{F_1^*}\Phi(r_{PQ})d\nu_0^*(P).$$

Hence we see $z^2\leqq zy$. It is impossible by Lemma 1 that $z^2 = zx$ and $z^2 = zy$ hold simultaneously. Thus, we obtain $G^* = \dfrac{xy}{z^2}>1$.

Lemma 5: *Let E_1 and E_2 be two disjoint bounded Borel sets of positive Φ-capacity. Then $G(\mu, \nu)\geqq 1$ for $\mu\in D_{E_1}$ and $\nu\in D_{E_2}$.*

Proof: Let $\{F_n^{(1)}\}$ and $\{F_n^{(2)}\}$ be sequences of closed subsets of E_1 and E_2 respectively such that

$$F_1^{(1)}\subset F_2^{(1)}\subset F_3^{(1)}\subset\cdots\cdots\subset E_1,$$

$$F_1^{(2)}\subset F_2^{(2)}\subset F_3^{(2)}\subset\cdots\cdots\subset E_2,$$

$\mu(F_n^{(1)})\uparrow\mu(E_1)$ and $\nu(F_n^{(2)})\uparrow\nu(E_2)$; then we can easily see that $C(F_n^{(1)})>0$ and $C(F_n^{(2)})>0$ for sufficiently large n. By Lemma 4,

$$G_n(\mu, \nu) = \frac{\left(\dfrac{1}{\mu(F_n^{(1)})}\right)^2\iint_{F_n^{(1)}}\Phi(r_{PQ})d\mu(Q)d\mu(P)\times\left(\dfrac{1}{\nu(F_n^{(2)})}\right)^2\iint_{F_n^{(2)}}\Phi(r_{PQ})\,d\nu(Q)d\nu(P)}{\left[\dfrac{1}{\mu(F_n^{(1)})}\,\dfrac{1}{\nu(F_n^{(2)})}\int_{F_n^{(1)}}d\mu(P)\int_{F_n^{(2)}}\Phi(r_{PQ})d\nu(Q)\right]^2}$$

$$>1.$$

Making $n\to\infty$, we obtain $G(\mu,\nu)\geqq 1$.

Lemma 6: *Under the same condition as in Lemma 5,* $G(\mu,\nu)>1$ *for* $\mu\in D_{E_1}$ *and* $\nu\in D_{E_2}$.

Proof: Suppose that $G(\mu^*,\nu^*)=1$ for $\mu^*\in D_{E_1}$ and $\nu^*\in D_{E_2}$. Then (μ^*,ν^*) minimizes $G[\mu\in D_{E_1},\nu\in D_{E_2}]$. For μ^* and ν^*, let $x, y, z, g_1(P)$ and $g_2(P)$ be same as in Lemma 3, and

$$E_1' = \{P; P\in E_1, g_1(P)\leqq 0\} \quad\text{and}\quad E_1''=\{P; P\in E_1, g_1(P)>0\};$$

then evidently $z^2=xy$ and $\sqrt{y}\,g_1(P)+\sqrt{x}\,g_2(P)=0$. As

$$\mu^*\in D_{E_1'},\ \nu^*\in D_{E_2}\subset D_{E_2+E_1''}$$

by Lemma 3 and

$$G(\mu^*,\nu^*)=1,\ (\mu^*,\nu^*)$$

also minimizes

$$G\left[\mu\in D_{E_1'},\ \nu\in D_{E_2+E_1''}\right]$$

by Lemma 5. Hence, by Lemma 3

$$C\{P; P\in E_2+E_1'', g_2(P)<0\}=0,$$

and so

$$C\{P; P\in E_1'', g_2(P)<0\}=0.$$

Namely,

$$C\left\{P; P\in E_1'', \sqrt{\frac{y}{x}}\,g_1(P)>0\right\}=0.$$

Thus, $C(E_1'')=0$.

Accordingly,

$$C\{P; P\in E_1, g_1(P)\neq 0\}=0.$$

Similarly, we see

$$C\{P; P\in E_2, g_2(P)\neq 0\}=0$$

and so

$$C\left\{P; P\in E_2, \sqrt{\frac{y}{x}}\,g_1(P)\neq 0\right\}=0.$$

Thus we obtain $g_1(P)\cong 0$ in E_1+E_2. Next we shall show that $g_1(P)\cong 0$ in $\overline{E_1+E_2}$. Let M be any bounded Borel set contained in $\overline{E_1+E_2}-(E_1+E_2)$. As $\mu^*\in D_{E_1}$, $\nu^*\in D_{E_2+M}$ and $G(\mu^*,\nu^*)=1$, (μ^*,ν^*) also minimizes $G[\mu\in D_{E_1},\nu\in D_{E_2+M}]$ by Lemma 5. Hence,

$$C\{P;\, P\in E_2+M,\, g_2(P)<0\} = 0$$

and so $\quad C\{P;\, P\in M,\, g_2(P)<0\} = 0.$

Therefore, $\quad C\left\{P;\, P\in M,\, \sqrt{\frac{y}{x}}\, g_1(P)>0\right\} = 0.$

Similarly, $\quad C\{P;\, P\in E_1+M,\, g_1(P)<0\} = 0,$

and so $\quad C\{P;\, P\in M,\, g_1(P)<0\} = 0.$

Thus, $\quad C\{P;\, P\in M,\, g_1(P)\neq 0\} = 0.$

As M may be taken arbitrarly in

$$\overline{E_1+E_2}-(E_1+E_2),$$

we obtain $g_1(P)\cong 0$ in $\overline{E_1+E_2}$. Let μ_0 be the equilibrium distribution on $\overline{E_1+E_2}$. Then

$$0 = \int_{\overline{E_1+E_2}} g_1(P)d\mu_0(P) = z^2 V_{\overline{E_1+E_2}} - zx V_{\overline{E_1+E_2}}$$

and so $z = x$.

Finally, $\quad \int_{E_1} \Phi(r_{PQ})d\mu^*(Q) \cong \int_{E_2} \Phi(r_{PQ})d\nu^*(Q)$

in $\overline{E_1+E_2}$, which contradicts to Lemma 2.

Corollary: *Under the same condition*

$$\frac{\iint_{E_1} \Phi(r_{PQ})d\mu(Q)\, d\mu(P) \times \iint_{E_2} \Phi(r_{PQ})d\nu(Q)d\nu(P)}{\left[\int_{E_1} d\mu(P)\int_{E_2} \Phi(r_{PQ})\, d\nu(Q)\right]^2} > 1$$

for $\mu\in D^m_{E_1}$ *and* $\nu\in D^n_{E_2}$, *where* m *and* n *are arbitrary positive numbers.*

As an immediate consequence of this corollary, we obtain

Theorem: *Let* $\Phi(t)\geqq 0$ *and satisfy the conditions* α) *and* β). *Then an energy integral*

$$I(\sigma) = \iint_E \Phi(_{PQ})d\sigma(Q)d\sigma(P)$$

with respect to any mass distribution σ *on a bounded Borel set* E, *if it exists, is always* $\geqq 0$; *especially the equality holds if and only if* $\sigma \equiv 0$.

Remark: In our proof, we can easily see that the non-negativity of $\Phi(t)$ has only to be supposed in $(0, \delta(E))$. Accordingly, we obtain the following corollary: *Let* $\Phi(t)$ *satisfy the conditions* α) *and* β). *Then an energy integral*

$$I(\sigma) = \iint_E \Phi(r_{PQ}) d\sigma(Q) d\sigma(P)$$

with respect to any mass distribution σ *of* algebraic 0 *on a bounded Borel set* E, *if it exists, is always* $\geqq 0$; *especially the equality holds if and only if* $\sigma \equiv 0$.