23. On a Sufficient Condition for a Tensor to be Harmonic

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We prove the following theorem, which is an extension of Besicovitch's theorems[1].

Theorem. *Let α be a p-form*[2] *on an n-dimensional Riemannian space* C^{∞},[2] *M, and* $a_{i_1 i_2 \dots i_p}$ *be its bounded coefficients which are defined and continuous on M except at most at the points of a set* E_1 *of* $(n-1)$-*dimensional measure* 0 *and further which is totally differentiable and satisfies*

$$d\alpha = \delta\alpha = 0$$

at every point except at most those of a set E_2 *expressible as the sum of an enumerable infinity of sets of finite* $(n-1)$-*dimensional measure; then* α *is harmonic* (*in Hodge's sense*) *on M.*

Lemma. *Suppose that F is a continuous additive function of an interval in the space* R_n, *such that* $F(I)/[\delta(I)]^a$ *is bounded and which fulfils the condition* (1_a) *at every point except at most those of a set* E_1^* *of measure* (Λ_a) 0, *where I is an arbitrary interval and* $0 \leqslant a < n$, *and that g is a summable function. Suppose further that* (i) $(\mathfrak{Q})\underline{F}(x) > -\infty$ *at every point x except at most those of a set* E_2^* *expressible as the sum of an enumerable infinity of sets of finite measure* (Λ_a), *and that* (ii) $(\mathfrak{Q})\underline{F}(x) \geqslant g(x)$ *at almost all points x; then*

$$F(I_0) \geqslant \int_{I_0} g(x)\, dx$$

for every interval I_0.

A particular case of this lemma is stated in Saks' "Theory of the integral", p. 193. This lemma can be proved quite similarly. Also for the notations and the terms used in it, see this book.

Proof of the theorem. We may clearly suppose that M is a domain D on R_n with the coordinate-system $x_1, x_2, \dots, x_n$,[3] and D contains the interval $I_1 : 0 \leqq x_1 \leqq 1,\ 0 \leqq x_2 \leqq 1, \dots, 0 \leqq x_n \leqq 1$, and further

1) A. S. Besicovitch: On sufficient conditions for a function to be analytic and on behaviour of analytic functions in the neighbourhood of non-isolated singular points, Proc. Lond. Math. Soc. (2) **32**, 1-9 (1931). The two theorems stated in this note can be brought to a combined theorem, which is a particular case of our theorem. Though this is not so difficult to see, perhaps none has remarked it. Besicovitch's original proofs are not applicable to this combined theorem.

2) See de Rham and Kodaira: Harmonic Integrals, Mimeographed Notes, Institute for Advanced Study, Princeton (1950).

3) The Riemannian metric defined on D is however, of course, not necessarily Euclidean.

that what is to be proved is that α is harmonic (in Hodge's sense) inside I_1.

Now let η be an arbitrary $C^\infty(p-1)$-form on R_n, whose carrier is contained in I_1. We define an additive function of an interval

$$\Phi(I) = (-1)^{np+n+1}\int_{I\cdot I_1} \alpha \wedge * d\eta + \int_{B(I\cdot I_1)} *\alpha \wedge \eta \,.$$

On account of Lebesgue's theorem, this is continuous, since the coefficients of α are bounded and E_1 is of $(n-1)$-dimensional measure 0. Next $\Phi(I)/[\delta(I)]^{n-1}$ is bounded, since the coefficients of α are bounded. $\Phi(I)$ satisfies the condition (1_{n-1}) at every point $\bar{x}$ except at most those of the set E_1, since by Stokes' theorem[4]

$$\Phi(I) = (-1)^{np+n+1}\int_{I\cdot I_1} (\alpha-\alpha(\bar{x})) < * d\eta + \int_{B(I\cdot I_1)} *(\alpha-\alpha(\bar{x}))\wedge \eta$$

and the coefficients of α are continuous at $\bar{x}$, where $\alpha(\bar{x})$ is the p-form whose coefficients are respectively the constant values assumed by those corresponding of α at $\bar{x}$. We have further $(\mathfrak{D})\Phi(\bar{x})=0$ at every point $(\bar{x}=(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_n))$ except at most at those of the set E_2. In fact, since at such points the coefficients of α are totally differentiable and both $\delta\alpha$ and $d\alpha$ vanish, we have by Stokes' theorem and by virtue of the harmonicity (in Hodge's sense) of $\alpha^*(\bar{x})$

$$\begin{aligned}
\Phi(I) &= (-1)^{np+n+1}\int_{I\cdot I_1} (\alpha-\alpha(\bar{x}) - \alpha^*(\bar{x})) \wedge * d\eta \\
&\quad + \int_{B(I\cdot I_1)} *(\alpha-\alpha(\bar{x}) - \alpha^*(\bar{x})) \wedge \eta \\
&\quad + (-1)^{np+n+1}\int_{I\cdot I_1} (\alpha(\bar{x}) + \alpha^*(\bar{x})) \wedge * d\eta + \int_{B(I\cdot I_1)} *(\alpha(\bar{x}) + \alpha^*(\bar{x})) \wedge \eta \\
&= (-1)^{np+n+1}\int_{I\cdot I_1} (\alpha-\alpha(\bar{x}) - \alpha^*(\bar{x})) \wedge * d\eta \\
&\quad + \int_{B(I\cdot I_1)} *(\alpha-\alpha(\bar{x}) - \alpha^*(\bar{x})) \wedge \eta \,,
\end{aligned}$$

where

$$\alpha^*(\bar{x}) = \sum_{i_1<i_2<\dots<i_p} \left[\sum_{k=1}^{n} \left\{\left(\frac{\partial a_{i_1 i_2\dots i_p}}{\partial x_k}\right)(x_k-\bar{x}_k)\right\} dx_{i_1}\wedge dx_{i_2}\wedge\dots\wedge dx_{i_p}\right],$$

and this implies $(\mathfrak{D})\underline{\Phi}(\bar{x})=0$. By the above lemma, we have therefore

$$\Phi(I_1) = \int_{I_1} \alpha \wedge * d\eta \geqq 0 \,.$$

By the similar consideration of the additive function of an interval $-\Phi(I)$, we have also

$$-\Phi(I_1) = -\int_{I_1} \alpha \wedge * d\eta \geqq 0 \,.$$

Thus we have

$$\int_{I_1} \alpha \wedge * d\eta = 0 \,. \qquad (1)$$

4) Loc. cit. 2).

By the similar consideration of the additive function of an interval

$$\Phi^*(I)=\int_{I\cdot I_1}\alpha\wedge*\delta\zeta+\int_{B(I\cdot I_1)}\alpha\wedge*\zeta$$

defined for an arbitrary $C^\infty(p+1)$-form ζ whose carrier is contained in I_1, as above, we have

$$\int_{I_1}\alpha\wedge*\delta\zeta=0\,. \tag{2}$$

In virtue of the theorem proved by Kodaira and de Rham[5], it follows immediately from (1) and (2) that α is harmonic inside I_1, q.e.d.

5) Loc. cit. 2).