# 64. On the Representations of Operator Algebras

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By the affirmative settlement of the Gelfand-Neumark conjecture, every $B^*$-algebra can be represented as a uniformly closed operator algebra on a suitable hilbert space.[1] However, though we have today considerable literatures concerning uniformly closed operator algebras, the theory of such algebras is fairly under developed than that of weakly closed ones. Thus the following problem arises naturally: *When does a $B^*$-algebra permit a representation as a weakly closed operator algebra?*[2] We discussed this problem in the preceding paper and obtained the condition in terms of the conjugate space of such an algebra.[3] But in the theory of operator algebras, state spaces are more preferred to conjugate spaces, hence it is more desirable to find the characteristic properties of state spaces for weakly closed operator algebras. The main aim of this paper is to get an answer for this question through the analysis of underlying spaces of operator algebras.

1. Let $A$ be a weakly closed self-adjoint operator algebra on a hilbert space $H$ and $f$ be a linear functional continuous with respect to the strongest topology of $A$, then $f$ can be expressed in the form:[4]

$$f(x)=\sum_{i=1}^{\infty}<x\varphi_i, \psi_i> \qquad \text{for } x\in A, \tag{1}$$

where $\varphi_i, \psi_i$ are elements of $H$ such as $\sum_{i=1}^{\infty}||\varphi_i||^2<+\infty$, $\sum_{i=1}^{\infty}||\psi_i||^2<+\infty$ and $<,>$ denotes the inner product of $H$.

**Lemma 1.** *A strongest continuous state $\sigma$ of $A$ i.e. a state which is continuous with respect to the strongest topology has an expression such as*

$$\sigma(x)=\sum_{i=1}^{\infty}<x\varphi_i, \varphi_i> \qquad \text{for } x\in A, \tag{2}$$

*where $\varphi_i$ are elements of $H$ satisfying* $\sum_{i=1}^{\infty}||\varphi_i||^2=1.$[5]

J. Dixmier has shown a short proof of this lemma in the recent paper: Sur les anneaux d'opérateur dans les espaces hilbertiens, C. R. Paris 238 (1954), No. 4. The same fact was pointed out by Prof. Fukamiya in conversation.

In the followings, *$C^*$-representation* of a $B^*$-algebra means a faithful representation as a uniformly closed operator algebra on a certain hilbert space and, when the latter algebra is weakly closed, we call it a *$W^*$-representation.* To avoid the complexity, in this note, we promise to use the same symbol for an element of $B^*$-algebra and its corresponding operator with respect to a $C^*$-representation. Then *a strongest continuous state with respect to a $C^*$-representation* is a state which permits an expression in the form (2) by elements of the underlying space of this $C^*$-representation. The state space $\Omega$ of a $B^*$-algebra $A$ is the collection of all states of $A$ which is topologized as a subset in the conjugate space $\bar{A}$ of $A$ by two methods. The first is the subtopology induced by the norm topology of $\bar{A}$ and the other is the one given by the weak topology as functionals of $\bar{A}$. Simply we call them *norm topology* and *weak topology* of $\Omega$ respectively. Let $\sigma$, $\tau$ be two states of $A$ and $\{H_\sigma, A_\sigma\}$, $\{H_\tau, A_\tau\}$ be representations of $A$ on $H_\sigma$ and $H_\tau$ constructed from $\sigma$ and $\tau$ respectively by the usual method. If there exists an invariant subspace $M$ in $H_\sigma$ such that the restriction $A_{\sigma(M)}$ of $A_\sigma$ to $M$ is unitarily equivalent to the representation $\{H_\tau, A_\tau\}$, we define an order for $\sigma$ and $\tau$ by $\sigma > \tau$. Let $S$ be a collection of states and $\{H_\sigma, A_\sigma\}$ be the usual representation of $A$ by $\sigma \in S$, the representation on direct sum of $H_\sigma$ $(\sigma \in S)$ which coincides with $\{H_\sigma, A_\sigma\}$ on each component space $H_\sigma$ is called simply *the representation on the direct sum of* $H_\sigma(\sigma \in S)$.

**Definition 1.** A *basic subset* $S$ of state space $\Omega$ is a set of states of $A$ satisfying the following conditions:

(i) $S$ is dense in $\Omega$ by the weak topology,

(ii) $S$ is closed by the norm topology of $\Omega$,

(iii) $S$ is convex,

(iv) if $\sigma \in S$ and $\sigma > \tau$ then $\tau \in S$.

Then we can get an inner characterization for the set of all strongest continuous states of a $C^*$-algebra as follows:

**Theorem 1.** *Let $S$ be a subset of the state space $\Omega$ of a $B^*$-algebra $A$. Then there exists a $C^*$-representation of $A$ such that $S$ coincides with the totality of strongest continuous states if and only if $S$ is a basic subset.*

**Proof.** Necessity. Clearly the totality $S$ of strongest continuous states with respect to a certain $C^*$-representation is convex and closed by norm topology. Assume $S$ is not dense in $\Omega$ and denote its closure in $\Omega$ by $\bar{S}$, then there exists a state $\sigma \in \Omega - \bar{S}$. As $S$ is regularly convex in the conjugate space of $A$, there exists a self-adjoint $a \in A$ such as $\sup_{s \in \bar{S}} s(a) < \sigma(a)$. By the spectral decomposition, $a$ is the difference of two non-negative operators i.e. $a = a_+ - a_-$.

We can assume $a_+>0$ without loss of generality, then

$$\|a_+\| = \sup_{\|\varphi\|=1,\varphi\in H} <a\varphi,\varphi> \leqq \sup_{s\in \bar{S}} s(a) < \sigma(a) \tag{3}$$
$$= \sigma(a_+)-\sigma(a_-) \leqq \sigma(a_+) \leqq \|a_+\|,^{6)}$$

where $H$ is the underlying space of the $C^*$-representation. This contradiction shows the denseness of $S$ in $\Omega$.

Let $H_\varphi$ be the subspace spanned by $\{a\varphi,\ a\in A\}$ for $\varphi\in H$. Then, if $\sigma(x)=\sum_{i=1}^{\infty}<x\varphi_i,\varphi_i>$, it is easy to prove that the representation $\{H_\sigma, A_\sigma\}$ of $A$ constructed by $\sigma\in S$ is unitarily equivalent to a representation on an invariant subspace in the direct sum of $H_{\varphi_i}$ $(i=1,2,\ldots)$. Hence, if $\sigma>\tau$ then $\tau$ is also a strongest continuous state.

**Sufficiency.** Let $S$ be a basic subset of $\Omega$. For each state $\sigma\in S$, we construct a representation $\{H_\sigma, A_\sigma\}$. Then by the denseness of $S$, the representation on the direct sum $H$ of $H_\sigma(\sigma\in S)$ is a $C^*$-representation.[3] By the definition of $H$ and the conditions of $S$, the state $\sigma$ defined by $\sigma(x)=<x\varphi,\varphi>$ or $\sigma(x)=\sum_{i=1}^{\infty}<x\varphi_i,\varphi_i>$, where $\varphi$ or $\varphi_i$ are elements of $H$ such as $\|\varphi\|=1$ or $\sum_{i=1}^{\infty}\|\varphi_i\|^2=1$, is contained in $S$. Hence the totality of strongest continuous states with respect to this $C^*$-representation coincides with $S$. q.e.d.

Next we investigate the underlying space of a general $C^*$-representation.

For a subset $T$ of the state space $\Omega$, by $[T]$ denotes the smallest subset in $\Omega$ which contains $T$ and satisfies the conditions (ii), (iii) and (iv) of Definition 1.

**Theorem 2.** *Given a basic subset $S$ in the state space of a $B^*$-algebra $A$. Then the set of all strongest continuous states of a $C^*$-representation of $A$ coincides with $S$ if and only if there exists a collection $T$ of states contained in $S$ satisfying $[T]=S$ and the $C^*$-representation is unitarily equivalent to the representation on the direct sum of $c_\tau$-fold copy of $H_\tau$ for $\tau\in T$, where $c_\tau$ are suitable non-zero cardinals.*[7]

**Proof.** Let $H$ be an underlying space of a $C^*$-representation of $A$, then we take up sufficiently many elements $\varphi_k(k\in K)$ of $H$ such that $\|\varphi_k\|=1$ and $H$ equals to the direct sum of mutually orthogonal subspace $H_k$ where $H_k$ is the subspace spanned by $\{a\varphi_k;\ a\in A\}$. Clearly the restriction of $A$ on $H_k$ is unitarily equivalent to the representation constructed by the state $\sigma_k(x)$ defined by $\sigma_k(x)=<x\varphi_k,\varphi_k>$. If representations constructed by $\sigma_\alpha(x)=<x\varphi_\alpha,\varphi_\alpha>(\alpha\in\Gamma$, where $\Gamma$ is a subset of $K$ and put $c_\alpha$ the cardinal of $\Gamma)$ are unitarily equivalent each other, the representations on the direct sum of these underlying spaces is unitarily equivalent to the representation on the $c_\alpha$-fold copy of one of the underlying spaces $H_\alpha$.

Put $T$ the maximal collection of $\sigma_k(k \in K)$ such that any pair of representations by these states are not unitarily equivalent each other. Then every state $\sigma$ defined by $\psi \in H$ such as $\|\psi\|=1$ is dominated by a $\tau \in [T]$, hence $\sigma \in [T]$ and so every strongest continuous state belongs to $[T]$ too. Thus $T$ has the property described in the theorem and the $C^*$-representation is clearly unitarily equivalent to the representation on the direct sum of $c_\tau$-fold copy of $H_\tau$ for $\tau \in T$, where $c_\tau$ is the cardinal determined by $\tau$ as above.

On the contrary, if $T$ satisfies the assumption of the theorem, we can construct the representation of $A$ on the direct sum of $c_\tau$-fold copy of $H_\tau(\tau \in T)$ where $c_\tau$ are arbitrary non-zero cardinals. If $a \in A$ is mapped to 0 by this representation, $\sigma(a)=0$ for every $\sigma \in S$, hence $a=0$. Thus this representation is a $C^*$-representation. Moreover, as $[T]=S$, $S$ is identical with the set of all strongest continuous states with respect to this $C^*$-representation. q.e.d.

2. In this section, we treat $W^*$-representations. *When does a $B^*$-algebra permit a $W^*$-representation?* We reply to this question in the following form.

**Theorem 3.** *A $B^*$-algebra $A$ is $W^*$-representable if and only if there exists a rudimentary subset in the state space $\Omega$.*

A *rudimentary subset* $S_0$ means a basic subset having the following property: Let $W$ be the weak closure of a $C^*$-representation of $A$ on the direct sum of all $H_\omega(\omega \in \Omega)$,[3] then $W$ considered as a linear space is the direct sum of $A$ and $S_0^\perp$ where $S_0^\perp$ is the set of all elements of $W$ such as $\sigma(w^*w)=0$ for all $\sigma \in S_0$.

**Proof.** Assume that $A$ is $W^*$-representable and $S_0$ is the set of all strongest continuous states with respect to a $W^*$-representation of $A$. Let $x_\alpha$ in $A$ converges to $w$ in $W$ in the strong topology and satisfies $\|x_\alpha\| \leqq \|w\|$, then $\sigma(x_\alpha)$ converges to $\sigma(w)$ for all $\sigma \in \Omega$. On the other hand, a subfamily $x_\beta$ of $x_\alpha$ converges to $x$ in $A$ by the weak topology of the $W^*$-representation of $A$. This shows that $\sigma(x_\beta)$ converges to $\sigma(x)$ for all $\sigma \in S_0$. Thus $\sigma(w-x)=0$ for all $\sigma \in S_0$. As $S_0 \supset [S_0]$, every state defined by an element of $H_\sigma(\sigma \in S_0)$ is contained in $S_0$ also. Hence $w-x$ is mapped to 0 by the representation of $W$ on $H_\sigma$ and this implies $\sigma((w-x)^*(w-x))=0$, i.e. $w-x \in S_0^\perp$. By the denseness of $S_0$ in $\Omega$, 0 is the only element of $A$ contained in $S_0^\perp$, which shows the direct decomposition of $W$ into $A$ and $S_0^\perp$.

Conversely, we assume that $W$ is decomposable into $A$ and $S_0^\perp$. Then the representation of $W$ on the direct sum of $H_\sigma(\sigma \in S_0)$ is weakly closed, and every element of $S_0^\perp$ is mapped to 0 by this representation. Hence the $C^*$-representation of $A$ on the direct sum of $H_\sigma(\sigma \in S_0)$ must be weakly closed. q.e.d.

Thus the rudimentary subset of a $W^*$-representable algebra is nothing but the totality of strongest continuous states with respect to a $W^*$-representation. On the other hand, J. Dixmier has proved the strongest topology of a $W^*$-algebra is purely algebraic,[4] hence the rudimentary subset is unique. We can introduce an order into the family of all basic subsets in the state space of a $B^*$-algebra by inclusion relation as sets. Then the rudimentary subset has the following property:

**Proposition.** *The rudimentary subset is a minimal basic set in the state space by the order defined as above.*

**Proof.** Let $S_0$ be the rudimentary subset and $S$ be a basic subset contained in $S_0$. If $S \neq S_0$, there exists a hermitian element $w \in W$ and $\sigma_0 \in S_0 - S$ such as

$$\sup_{\sigma \in S} \sigma(\omega) < \sigma_0(\omega).$$

By the definition of the rudimentary subset, $w$ is decomposed into $w = a + s_0^{\perp}$ where $a$, $s_0^{\perp}$ are hermitian elements of $A$ and $S_0^{\perp}$ respectively. Then we can easily get a contradiction similarly as (3).

**Theorem 4.** *If $A$ is a $B^*$-algebra representable as a $W^*$-algebra, the representation constructed for $T$ in Theorem 2 is a $W^*$-representation if and only if $[T]$ coincides with the rudimentary subset in state space of $A$.*

**Proof.** If a $W^*$-representation is constructed by $T$, $[T]$ is identical with the collection of all strongest continuous states with respect to this $W^*$-representation i.e. the rudimentary subset. On the other hand, when $T$ satisfies the conditions of the theorem, the $C^*$-representation constructed by $T$ is a $W^*$-representation. For let $W$ be the weak closure of the $C^*$-representation by $T$. Then, both $W$ and $A$ considered as Banach spaces are conjugate spaces of the same Banach space composed of all finite linear combinations of states in the rudimentary subset and moreover there exists a canonical isometric mapping $A$ into $W$, thus $W$ must be identical with $A$, that is, the $C^*$-representation must be a $W^*$-representation. q.e.d.

As a consequence of this theorem the multiplicity theory of the commutative $W^*$-algebra can be easily explained. Besides, this theorem shows the reason why the double conjugate space of all completely continuous operators on a hilbert space can be isomorphic with the totality of bounded operators on the *same* space, which was not replied from the result of the preceding paper.[3]

## References

1) Cf. M. Fukamiya: On a theorem of Gelfand and Neumark and the $B^*$-algebra, Kumamoto Journ., **1**, 17-22 (1952), Math. Review, **14**, 884 (1953).

2) Cf. I. Kaplansky: Projections in Banach algebras, Ann. Math., **53**, 235-249 (1951).

3) Cf. Z. Takeda: Conjugate spaces of operator algebras, Proc. Japan Acad., **30**, 90-95 (1954).

4) Cf. J. Dixmier: Formes linéaires sur un anneau d'opérateurs, Bull. Soc. Math. France, **81**, 9-39 (1953).

5) This lemma was conjectured by J. Dixmier in loc. cit.

6) The author was suggested this inequality in a letter from H. Umegaki.

7) We neglect the trivial subspace on which every element of $A$ is represented to the 0-operator.