75. Ergodic Decomposition of Stationary Linear Functional*)

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In this note, we shall prove ergodic decomposition of stationary semi-trace of a separable D^*-algebra with a motion, applying the reduction theory of von Neumann [2][1] and a decomposition of a two-sided representation [3]. The theorem in this paper contains the ergodic decompositions of stationary trace on separable C^*-algebra with a motion and the ergodic decomposition of invariant regular measure on separable locally compact Hausdorff space with a group of homeomorphisms. (Cf. Th. 4 and Th. 7 of [3].)

Let $\mathfrak{A}$ be a D^*-algebra (: normed $*$-algebra over the complex number field) with an approximate identity $\{e_a\}$ and with a motion G where G is meant by any group of isometric *automorphisms on $\mathfrak{A}$. (Cf. [3].) Let τ be a G-stationary semi-trace of $\mathfrak{A}$, i.e. τ is a linear functional on $\mathfrak{A}^2$ (=self-adjoint (s.a.) subalgebra generated by $\{xy\,;\,x,\,y\in\mathfrak{A}\}$) such that $\tau(x^*x)\geqq 0$, $\tau(xy)=\tau(yx)=\overline{\tau(x^*y^*)}$, $\tau((xy)^*xy)\leqq\|x\|^2\tau(y^*y)$, $\tau((e_ax)^*e_ax)\xrightarrow[a]{}\tau(x^*x)$ and $\tau(x^sy^s)=\tau(xy)$ for all $x,\,y\in\mathfrak{A}$ and $s\in G$. Putting $\mathfrak{N}=\{x\,;\,\tau(x^*x)=0,\ x\in\mathfrak{A}\}$, $\mathfrak{N}$ is a two-sided ideal in $\mathfrak{A}$. Let $\mathfrak{A}^\theta$ be the quotient algebra $\mathfrak{A}/\mathfrak{N}$ and x^θ the class ($\in\mathfrak{A}^\theta$) containing x which is an incomplete Hilbert space with inner product $(x^\theta,\,y^\theta)=\tau(y^*x)$. Let $\mathfrak{H}$ be the completion of $\mathfrak{A}^\theta$ with respect to the norm $\|y^\theta\|\,(=\tau(y^*y)^{1/2})$. Putting $x^ay^\theta=(xy)^\theta$, $x^by^\theta=(yx)^\theta$, $jy^\theta=y^{*\theta}$ and $U_sy^\theta=y^{s\theta}$ for all $x,\,y\in\mathfrak{A}$ and $s\in G$, $\{x^a,\,x^b,\,j,\,\mathfrak{H}\}$ defines a two-sided representation of $\mathfrak{A}$. (Cf. [3].) Moreover $\{U_s,\,\mathfrak{H}\}$ defines a dual unitary representation of G. Indeed, for any $x,\,y\in\mathfrak{A}$ $(U_sy^\theta,\,U_sy^\theta)=(x^{s\theta},\,y^{s\theta})=\tau(y^sx^{*s})=(y^\theta,\,x^\theta)$ and $U_{st}y^\theta=y^{st\theta}=U_ty^{s\theta}=U_tU_sy^\theta$. Hence U_s has uniquely unitary extension on $\mathfrak{H}$ which satisfies the required relations. These representations are uniquely determined by the given τ within unitary equivalence. (Cf. [3].)

For any collection F of bounded operators and two W^*-algebras W_1, W_2 on a Hilbert space, we denote F' the collection of all bounded operators commuting for all $A\in F$ and $W_1\smile W_2$ the W^*-algebra generated by W_1 and W_2.

Let W^a, W^b and W_G be W^*-algebras generated by $\{x^a;\,x\in\mathfrak{A}\}$, $\{x^b;\,x\in\mathfrak{A}\}$ and $\{U_s;\,s\in G\}$ respectively, then $W^a=W^{b\prime}$ and $jAj=A^*$ for all $A\in W^a\frown W^b$. (Cf. Th. 2 of [3].)

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*) This paper is a continuation of the previous paper [3].

1) Numbers in brackets refer to the references at the end of this paper.

A G-stationary semi-trace τ is called *G-ergodic*, if τ is not positively linear combination of any other linearly independent G-stationary semi-traces of $\mathfrak{A}$. Then τ is G-ergodic if and only if $\{x^a, x^b, U_s, j, \mathfrak{H}\}$ is irreducible, i.e. $W^a \frown W^b \frown W'_G = \{\lambda I\}$. (Cf. Th. 5 of [3].)

Let $\mathfrak{A}$ be separable with the motion G, then we have

Lemma 1. G contains an enumerable subset $\{s_n\}$ such that for any $x \in \mathfrak{A}$ and $t \in G$ there exists $\{t_n\} \subset \{s_n\}$ satisfying

$$\| x^{t_n} - x^t \| \to 0 \ (n \to \infty).$$

The proof of this lemma follows from the similar way in the proof of Th. 5 of [3].

In the following, we assume that $\mathfrak{A}$ is a separable D^*-algebra with G and has a G-stationary semi-trace τ, and moreover $\mathfrak{A}$ satisfies that there is an enumerable subset $\mathfrak{A}_c$ in $\mathfrak{A}$ with the property that: for any $x \in \mathfrak{A}$ there exists $y_x \in \mathfrak{A}_c$ (dependently on x) such that

$$(1) \qquad x = xy_x.$$

THEOREM. *For the D^*-algebra $\mathfrak{A}$, there exists a system of G-ergodic semi-traces π_λ such that*

$$(2) \qquad \tau(x) = \int \pi_\lambda(x) d\sigma(\lambda) \qquad \text{for all } x \in \mathfrak{A}$$

where λ runs over the whole real line R and the weight function $\sigma(\lambda)$ is an N-function in the sense of von Neumann. (Cf. [2].)

First we shall prove the following lemma:

Lemma 2. (i) *Any semi-trace ω of $\mathfrak{A}$ satisfies that for all x and $z \in \mathfrak{A}$*

$$| \omega(xz) | \leqq \| x \| \, \omega(z^*z)^{1/2} \omega(y_z^* y_z)^{1/2}.$$

(ii) *If $\omega(x^{*s_n} x^{s_n}) = \omega(x^*x)$ for all $x \in \mathfrak{A}$ and $n = 1, 2, \dots$, then ω is G-stationary.*

Proof. (i): $| \omega(xz) | = | \omega(xzy_z) | \leqq \omega((xz)^* xz)^{1/2} \, \omega(y_z^* y_z)^{1/2} \leqq \| x \| \, \omega(z^*z)^{1/2} \omega(y_z^* y_z)^{1/2}$. (ii): For $x \in \mathfrak{A}$ and $t \in G$, taking $\{t_n\} \subset \{s_n\}$ such that $\| x^{t_n} - x^t \| \to 0 \ (n \to \infty)$, $| \omega(x^{*t} x^t) - \omega(x^{*t_n} x^t) | = | \omega((x^{*t_n} - x^{*t}) x^t) | \leqq \| x^{t_n} - x^t \| \cdot M \to 0 \ (n \to \infty)$ and

$$\begin{aligned} | \omega(x^{*t_n} x^t) - \omega(x^{*t_n} x^{t_n}) | &= | \omega(x^{*t_n}(x^t - x^{t_n})) | \\ &\leqq \| x^t - x^{t_n} \| \cdot \omega(x^{*t_n} x^{t_n})^{1/2} \omega(y_x^{*t_n} y_x^{t_n})^{1/2} \\ &= \| x^t - x^{t_n} \| \cdot \omega(x^*x)^{1/2} \cdot \omega(y_x^* y_x)^{1/2} \to 0 \ (n \to \infty). \end{aligned}$$

Since $\omega(x^{*t_n} x^{t_n}) = \omega(x^*x)$, $\omega(x^{*t} x^t) = \omega(x^*x)$. As any $x \ (\in \mathfrak{A}) = xy_x = [(x + y_x)^*(x + y_x) + \cdots]/4$, $\omega(x^t) = \omega(x)$ for all $x \in \mathfrak{A}$ and $t \in G$.

Proof of THEOREM.[2] Let G_0 be subgroup of G generated by

2) Since $U_s x U_{s^{-1}} y^0 = x^{sa} y^0$ for all $x, y \in \mathfrak{A}$, putting $x^{as} = x^{sa}$, $x^a \to x^{as}$ is uniquely extended to a *automorphism on the C^*-algebra $\mathfrak{R}$ generated by $A^a = \{x^a;\ x \in \mathfrak{A}\}$ such that $A \in \mathfrak{R} \to A^s (= U_s A U_{s^{-1}}) \in \mathfrak{R}$, and G induces a motion on $\mathfrak{R}$. Since $\mathfrak{R}$ is separable and $\| A^s \| = \| U_s A U_{s^{-1}} \| = \| A \|$ for all $A \in \mathfrak{R}$ and $s \in G$, Lem. 1 for $(\mathfrak{R}, G)$ also holds. Considering the stationary semi-trace π_λ on $\mathfrak{A}^a$ with respect to the operator norm (in the place of $\mathfrak{A}$), the proof of this theorem may be possible without the assumption $\| x^s \| = \| x \| \ (x \in \mathfrak{A},\ s \in G)$.

$\{s_n\}$ and $\{x_n\}$ dense subset of $\mathfrak{A}$. Let $\mathfrak{A}_0$ be countable s.a. subring in $\mathfrak{A}$ generated by $\{x_n^s, y^t; s, t \in G_0, y \in \mathfrak{A}_c, n=1, 2, \dots\}$. Let $\mathfrak{A}_1$ be a C^*-algebra generated by $\{x^a, y^b, U_s; x, y \in \mathfrak{A}, s \in G_0\}$ which is obviously separable (in the uniform topology) and contains I. Putting $\boldsymbol{M} = W^a \frown W^b \frown W'_{G_0}$, $\boldsymbol{M}$ is a commutative W^*-algebra. Since $\mathfrak{H}$ is separable (cf. Lem. 6 of [3]), there exists a direct decomposition in the sense of von Neumann: $\mathfrak{H} = \int \mathfrak{H}_\lambda d\sigma(\lambda)$ and $A \sim \int A_\lambda \, (A \in \boldsymbol{M}')$ with respect to $\boldsymbol{M}$, where the N-function $\sigma(\lambda)$ is determined by M. Since $\mathfrak{A}'_1 = (W^a \smile W^b \smile W_{G_0})' = W^b \frown W^a \frown W'_{G_0}$, there exists a $\sigma(\lambda)$-null set $N_1 (\subset R)$ such that $\{A_\lambda; A \in \mathfrak{A}_1\}' = \{\alpha I_\lambda\}$ for $\lambda \bar{\in} N_1$, and since x^a, x^b, $U_s \in \boldsymbol{M}' \, (s \in G_0)$ they are decomposable:

$$x^a \sim \int x^{a(\lambda)}, \quad x^b \sim \int x^{b(\lambda)} \text{ and } U_s \sim \int U_s(\lambda).$$

Putting $y^\theta = \int y^{\theta(\lambda)}$, by Lem. 4 of [3] $\{x^{a(\lambda)}, x^{b(\lambda)}, j_\lambda, \mathfrak{H}_\lambda\}$ is a two-sided representation of $\mathfrak{A}$ and $U_s(\lambda)(s \in G_0)$ are unitary on $\mathfrak{H}_\lambda$ such that $U_{st}(\lambda) = U_t(\lambda) U_s(\lambda)$, $U_{s^{-1}}(\lambda) = U_s(\lambda)^{-1}$, $U_s(\lambda) + U_t(\lambda) = (U_s + U_t)(\lambda)$ and $U_s(\lambda) y^{\theta(\lambda)} = y^{s\theta(\lambda)}$ for all $s \in G_0$ and all $y \in \mathfrak{A}_0$ excepting a $\sigma(\lambda)$-null set N_2. Then $\{x^{a(\lambda)}, y^{b(\lambda)}, U_s(\lambda); x, y \in \mathfrak{A}_0, s \in G_0\}$ are irreducible for $\lambda \bar{\in} N_1 \smile N_2$. Since $\mathfrak{A}_0$ is countable, we can find a $\sigma(\lambda)$-null set N_3 such that

$$(x+y)^{\theta(\lambda)} = x^{\theta(\lambda)} + x^{\theta(\lambda)}, \ (xy)^{\theta(\lambda)} = x^{a(\lambda)} y^{\theta(\lambda)} = y^{b(\lambda)} x^{\theta(\lambda)} \text{ and } j_\lambda y^{\theta(\lambda)} = y^{*\theta(\lambda)}$$

for all $x, y \in \mathfrak{A}_0$ and $\lambda \bar{\in} N_3$.

Let $W^{a(\lambda)}$, $W^{b(\lambda)}$ and $W_G(\lambda)$ be W^*-algebras generated by $\{x^{a(\lambda)}; x \in \mathfrak{A}_0\}$, $\{x^{b(\lambda)}; x \in \mathfrak{A}_0\}$ and $\{U_s(\lambda); s \in G_0\}$ $(\lambda \bar{\in} N = \cup_i N_i)$ respectively. Because the closed linear extension $\mathfrak{M}$ (in $\mathfrak{H}_\lambda$ for $\lambda \bar{\in} N$) of $\{(xy)^{\theta(\lambda)}; x, y \in \mathfrak{A}_0\}$ is invariant under $x^{a(\lambda)}$, $y^{b(\lambda)}(x, y \in \mathfrak{A}_0)$, j_λ and $U_s(\lambda)(s \in G_0)$, $\mathfrak{M} = \mathfrak{H}_\lambda$, and $W^{a(\lambda)}$ and $W^{b(\lambda)}$ are weak closures of $\{x^{a(\lambda)}; x \in \mathfrak{A}\}$ and $\{x^{b(\lambda)}; x \in \mathfrak{A}\}$ respectively. (Cf. Lem. 1 of [3].)

Now we shall prove that $\mathfrak{H}_\lambda$ (for arbitrary, but fixed $\lambda \bar{\in} N$) is H-system. (Cf. [1].)

i) A vector $v \in \mathfrak{H}_\lambda$ is called bounded, if $\|x^{a(\lambda)} v\| \leqq M_v \|x^{\theta(\lambda)}\|$ for all $x \in \mathfrak{A}_0$ and a constant $M_v > 0$. Denote $\mathfrak{B}_\lambda$ the collection of all such $v \in \mathfrak{H}_\lambda$. It is evident that $\{x^{\theta(\lambda)}; x \in \mathfrak{A}_0\} \subset \mathfrak{B}_\lambda$. For any $v \in \mathfrak{B}_\lambda$, putting $v_\lambda^{b\prime} x^{\theta(\lambda)} = x^{a(\lambda)} v$ for all $x \in \mathfrak{A}_0$, $v^{b\prime}$ has unique bounded extension v^b which belongs to $W^{b(\lambda)}$. For, $x^{a(\lambda)} v^b y^{\theta(\lambda)} = x^{a(\lambda)} y^{a(\lambda)} v = v^b (xy)^{\theta(\lambda)} = v^b x^{a(\lambda)} y^{\theta(\lambda)}$ for all $x, y \in \mathfrak{A}_0$, and we can choose the $\sigma(\lambda)$-null set N such that $W^{b(\lambda)} = W^{a(\lambda)\prime}$ for $\lambda \bar{\in} N$.

ii) For $v \in \mathfrak{B}$, $j_\lambda v \in \mathfrak{B}_\lambda$ and $(j_\lambda v)^b = v^{b*}$. Indeed, $(x^{a(\lambda)} j_\lambda v, y^{\theta(\lambda)}) = (j_\lambda v, x^{*a(\lambda)} y^{\theta(\lambda)}) = (j_\lambda (x^* y)^{\theta(\lambda)}, v) = ((y^* x)^{\theta(\lambda)}, v) = (y^{*a(\lambda)} x^{\theta(\lambda)}, v) = (x^{\theta(\lambda)}, y^{a(\lambda)} v) = (x^{\theta(\lambda)}, v^b y^{\theta(\lambda)}) = (v^{b*} x^{\theta(\lambda)}, y^{\theta(\lambda)})$ for all $x, y \in \mathfrak{A}_0$ and hence $x^{a(\lambda)} j_\lambda v = v^{b*} x^{\theta(\lambda)}$.

iii) If $v \in \mathfrak{B}_\lambda (\lambda \bar{\in} N)$, then $\|x^{b(\lambda)} v\| \leqq M \|x^{\theta(\lambda)}\|$ for all $x \in \mathfrak{A}_0$ where M is a constant. For, $x^{b(\lambda)} v = j_\lambda x^{*a(\lambda)} j_\lambda v = j_\lambda (j_\lambda v)^b x^{*\theta(\lambda)} = j_\lambda v^{b*} x^{*\theta(\lambda)}$ and hence $\|x^{b(\lambda)}v\|=\|v^{b*}x^{*\theta(\lambda)}\|\leqq\|v^{b*}\|\cdot\|x^{*\theta(\lambda)}\|=\|v^b\|\cdot\|x^{\theta(\lambda)}\|$.

Putting $v^{a\prime}x^{\theta(\lambda)}=x^{b(\lambda)}v$ for all $x\varepsilon\mathfrak{A}_0$, $v^{a\prime}$ has a bounded extension v^a in $W^{a(\lambda)}$ such that $(j_\lambda v)^a=v^{a*}=j_\lambda v^b j_\lambda$. Proving only the last equation: $v^{a*}x^{\theta(\lambda)}=(j_\lambda v)^a x^{\theta(\lambda)}=x^{b(\lambda)}j_\lambda v=j_\lambda j_\lambda x^{b(\lambda)}j_\lambda v=j_\lambda v^b x^{*\theta(\lambda)}=j_\lambda v^b j_\lambda x^{\theta(\lambda)}$.

iv) $\mathfrak{B}_\lambda^a(=\{v^a;\ v\varepsilon\mathfrak{B}_\lambda\})$ and $\mathfrak{B}_\lambda^b(=\{v^b;\ v\varepsilon\mathfrak{B}_\lambda\})$ are two-sided ideals in $W^{a(\lambda)}$ and $W^{b(\lambda)}$ respectively. Since for any $v\varepsilon\mathfrak{B}_\lambda$ and $A\varepsilon W^{a(\lambda)}$ $x^{b(\lambda)}Av=Ax^{b(\lambda)}v=Av^a x^{\theta(\lambda)}$, $Av\varepsilon\mathfrak{B}_\lambda$ and $(Av)^a=Av^a$. Since $(j_\lambda v)^a=v^{a*}$, $\mathfrak{B}_\lambda^a$ is s.a. and hence a two-sided ideal in $W^{a(\lambda)}$. The case of $\mathfrak{B}_\lambda^b$ follows similarly.

v) For any $x\varepsilon\mathfrak{A}$ and $y\varepsilon\mathfrak{A}_0$, there exists uniquely $v\varepsilon\mathfrak{B}_\lambda$ such that $(xy)^{a(\lambda)}z^{\theta(\lambda)}=z^{b(\lambda)}v$ for all $z\varepsilon\mathfrak{A}_0$. For, by iv) $(xy)^{a(\lambda)}=x^{a(\lambda)}y^{a(\lambda)}$ belongs to $\mathfrak{B}_\lambda^a$ and hence we can find $v\varepsilon\mathfrak{B}_\lambda$ in the required relation. If $v_1, v_2\varepsilon\mathfrak{B}_\lambda$ satisfy $(xy)^{a(\lambda)}z^{\theta(\lambda)}=z^{b(\lambda)}v_1=z^{b(\lambda)}v_2$ for all $z\varepsilon\mathfrak{A}_0$, then $B_{v_1}=B_{v_2}$ for all $B\varepsilon W^{b(\lambda)}$ and hence $v_1=v_2$ in $\mathfrak{H}_\lambda$.

Denote $(xy)^{\varphi(\lambda)}$ the v corresponding to $x\varepsilon\mathfrak{A}$, $y\varepsilon\mathfrak{A}_0$. Then

(3) $$(xy)^{\varphi(\lambda)}=x^{a(\lambda)}y^{\theta(\lambda)}\qquad\text{for } x\varepsilon\mathfrak{A},\ y\varepsilon\mathfrak{A}_0.$$

For, $z^{b(\lambda)}(xy)^{\varphi(\lambda)}=x^{a(\lambda)}y^{a(\lambda)}z^{\theta(\lambda)}=x^{a(\lambda)}z^{b(\lambda)}y^{\theta(\lambda)}=z^{b(\lambda)}x^{a(\lambda)}y^{\theta(\lambda)}$ for all $z\varepsilon\mathfrak{A}_0$.

Similarly $(yx)^{\varphi(\lambda)}$ (for $y\varepsilon\mathfrak{A}_0$, $x\varepsilon\mathfrak{A}$) is well defined in $\mathfrak{B}_\lambda$: $z^{a(\lambda)}(yx)^{\varphi(\lambda)}=(yx)^{b(\lambda)}z^{\theta(\lambda)}$ for all $z\varepsilon\mathfrak{A}_0$. Then

(4) $$(yx)^{\varphi(\lambda)}=x^{b(\lambda)}y^{\theta(\lambda)}\qquad\text{for } x\varepsilon\mathfrak{A},\ y\varepsilon\mathfrak{A}_0.$$

For $z^{a(\lambda)}(yx)^{\varphi(\lambda)}=(yx)^{b(\lambda)}z^{\theta(\lambda)}=x^{b(\lambda)}y^{b(\lambda)}z^{\theta(\lambda)}=x^{b(\lambda)}z^{a(\lambda)}y^{\theta(\lambda)}=z^{a(\lambda)}x^{b(\lambda)}y^{\theta(\lambda)}$ for all $z\varepsilon\mathfrak{A}_0$.

(5) $$(x^*y^*)^{\varphi(\lambda)}=j_\lambda(yx)^{\varphi(\lambda)}\qquad\text{for all } x\varepsilon\mathfrak{A} \text{ and } y\varepsilon\mathfrak{A}_0.$$

For, $j_\lambda(x^*y^*)^{\varphi(\lambda)}=j_\lambda x^{*a(\lambda)}j_\lambda y^{\theta(\lambda)}=x^{b(\lambda)}y^{\theta(\lambda)}=(yx)^{\varphi(\lambda)}$.

vi) For any $x\varepsilon\mathfrak{A}_0$, $x^{\varphi(\lambda)}=x^{\theta(\lambda)}$ and $x^{\varphi(\lambda)}$ is uniquely determined. For, taking y_x in $\mathfrak{A}_0$, $x^{\theta(\lambda)}=(xy_x)^{\theta(\lambda)}=x^{a(\lambda)}y_x^{\theta(\lambda)}=(xy_x)^{\varphi(\lambda)}=x^{\varphi(\lambda)}$.

vii) For any $x, x_1\varepsilon\mathfrak{A}$, $x^{a(\lambda)}\varepsilon\mathfrak{B}_\lambda^a$ and $(x_1x)^{\varphi(\lambda)}=x_1^{a(\lambda)}x^{\varphi(\lambda)}=x^{b(\lambda)}x_1^{\varphi(\lambda)}$, and $x^{\varphi(\lambda)}$ is uniquely determined. This follows from the assumption (1), v), vi) and the (3), (4).

viii) $x^{*\varphi(\lambda)}=j_\lambda x^{\varphi(\lambda)}$ for all $x\varepsilon\mathfrak{A}$. For, taking the y_x as $x=xy_x$, $x^*=y_x^*x^*$, $x^{*\varphi(\lambda)}=(y_x^*x^*)^{\varphi(\lambda)}=j_\lambda(xy_x)^{\varphi(\lambda)}=j_\lambda x^{\varphi(\lambda)}$.

ix) Putting $\pi_\lambda(\sum_{k=1}^n x_ky_k)=\sum_{k=1}^n(x_k^{\varphi(\lambda)},\ j_\lambda y_k^{\varphi(\lambda)})$, $\pi_\lambda(\cdot)$ is well defined on $\mathfrak{A}^2$, and it is G-stationary semi-trace.

Indeed, if $\sum_{k=1}^n x_ky_k=\sum_{i=1}^m x_i'y_i'$, then for any $z\varepsilon\mathfrak{A}_0$

$$\begin{aligned}\sum_{k=1}^n(z^{a(\lambda)}x_k^{\varphi(\lambda)},\ j_\lambda y_k^{\varphi(\lambda)})&=\sum(x_k^{\varphi(\lambda)},\ z^{*a(\lambda)}y_k^{*\varphi(\lambda)})=\sum(x_k^{\varphi(\lambda)},\ y_k^{*b(\lambda)}z^{*\varphi(\lambda)})\\&=\sum(y_k^{b(\lambda)}x_k^{\varphi(\lambda)},\ z^{*\varphi(\lambda)})=(\sum(x_ky_k)^{\varphi(\lambda)},\ j_\lambda z^{\varphi(\lambda)})\\&=(\sum_{i=1}^m(x_i'y_i')^{\varphi(\lambda)},\ j_\lambda z^{\varphi(\lambda)})=\sum(z^{a(\lambda)}x_i'^{\varphi(\lambda)},\ j_\lambda y_i'^{\varphi(\lambda)}).\end{aligned}$$

Taking $\{z_\beta\}\subset\mathfrak{A}_0$ such that $z_\beta^{a(\lambda)}\to I$ (weakly), $\sum(x_k^{\varphi(\lambda)},\ j_\lambda y_k^{\varphi(\lambda)})=\sum(x_i'^{\varphi(\lambda)},\ j_\lambda y_i'^{\varphi(\lambda)})$. Hence $\pi_\lambda(\sum x_ky_k)=\pi_\lambda(\sum x_i'y_i')$. For any $x, y\varepsilon\mathfrak{A}$, $\pi_\lambda(xy)=(y^{\varphi(\lambda)},\ j_\lambda x^{\varphi(\lambda)})=(x^{\varphi(\lambda)},\ j_\lambda y^{\varphi(\lambda)})=\pi_\lambda(yx)$ and $\pi_\lambda((xy)^*xy)=\|(xy)^{\varphi(\lambda)}\|^2=\|x^{a(\lambda)}y^{\varphi(\lambda)}\|^2$

$\leqq \|x^{a(\lambda)}\|^2 \cdot \|y^{\varphi(\lambda)}\|^2 \leqq \|x\|^2 \cdot \pi_\lambda(y^*y)$. Taking $\{z_\beta\} \subset \mathfrak{A}_0$ such that $z_\beta^{a(\lambda)} \xrightarrow[\beta]{} I$ (strongly), $\|z_\beta^{a(\lambda)} x^{\varphi(\lambda)} - x^{\varphi(\lambda)}\| \xrightarrow[\beta]{} 0$. Hence $\|x^{\varphi(\lambda)} - e_\alpha^{a(\lambda)} x^{\varphi(\lambda)}\| \leqq \|x^{\varphi(\lambda)} - z_\beta^{a(\lambda)} x^{\varphi(\lambda)}\| + \|(z_\beta x)^{\varphi(\lambda)} - e_\alpha^{a(\lambda)} z_\beta^{a(\lambda)} x^{\varphi(\lambda)}\| + \|e_\alpha^{a(\lambda)} z_\beta^{a(\lambda)} x^{\varphi(\lambda)} - e_\alpha^{a(\lambda)} x^{\varphi(\lambda)}\| \leqq \|x^{\varphi(\lambda)} - z_\beta^{a(\lambda)} x^{\varphi(\lambda)}\| + \|z_\beta - e_\alpha z_\beta\| \cdot \|x^{\varphi(\lambda)}\| + \|e_\alpha\| \cdot \|z_\beta^{a(\lambda)} x^{\varphi(\lambda)} - x^{\varphi(\lambda)}\|$ and for any $\varepsilon > 0$ there exists α_0 such that $\|x^{\varphi(\lambda)} - e_\alpha^{a(\lambda)} x^{\varphi(\lambda)}\| < \varepsilon$ for $\alpha > \alpha_0$ or $\pi_\lambda((e_\alpha x)^* e_\alpha x) \xrightarrow[\alpha]{} \pi_\lambda(x^*x)$. Therefore $\pi_\lambda(\cdot)$ $(\lambda \bar{\varepsilon} N)$ are semi-traces of $\mathfrak{A}$. For and $x, y \varepsilon \mathfrak{A}$ and $s \varepsilon G_0$ $\pi_\lambda(x^s y^s) = (y^{s\varphi(\lambda)}, x^{*s\varphi(\lambda)}) = (U_s(\lambda) y^{\varphi(\lambda)}, U_s(\lambda) x^{*\varphi(\lambda)}) = (y^{\varphi(\lambda)}, x^{*\varphi(\lambda)}) = \pi_\lambda(xy)$. By Lem. 2 $\pi_\lambda(\cdot)$, $\lambda \bar{\varepsilon} N$, are G-stationary and we have ix).

For any $x \varepsilon \mathfrak{A}$ taking y_x in $\mathfrak{A}_0$, $\pi_\lambda(x^s) = \pi_\lambda(x^s y_x^s) = \pi_\lambda(x y_x) = \pi_\lambda(x)$ for any $s \varepsilon G$. Putting $U_s'(\lambda) x^{\theta(\lambda)} = x^{s\theta(\lambda)}$ for all $x \varepsilon \mathfrak{A}$, $U_s'(\lambda)$ has uniquely unitary extension $U_s(\lambda)$ which defines a dual unitary representation of G containing the dual ones of G_0.

These representations $\{x^{a(\lambda)}, x^{b(\lambda)}, j_\lambda, \mathfrak{H}_\lambda\}$ of $\mathfrak{A}$ and $\{U_s(\lambda), \mathfrak{H}_\lambda\}$ of G are corresponding to the stationary semi-traces $\pi_\lambda(\lambda \bar{\varepsilon} N)$. Since $W_0^{(\lambda)} = \{x^{a(\lambda)}, y^{b(\lambda)}, U_s(\lambda);\ x, y \varepsilon \mathfrak{A}, s \varepsilon G_0\}$, $\lambda \bar{\varepsilon} N$, are irreducible on $\mathfrak{H}_\lambda$, $W_0^{(\lambda)\prime} = (W^{a(\lambda)} \smile W^{b(\lambda)} \smile W_{G_0}(\lambda))' = \{\alpha I_\lambda\} = W^{b(\lambda)} \frown W^{a(\lambda)} \frown W_G(\lambda)'$. Therefore $\pi_\lambda(\cdot)$, $\lambda \varepsilon N$, are G-ergodic semi-traces.

For any $x \varepsilon \mathfrak{A}$ taking $y_x \varepsilon \mathfrak{A}_0$, $\tau(x) = \tau(x y_x) = (x^\theta, y_x^{*\theta}) = \int (x^{\theta(\lambda)}, y_x^{*\theta(\lambda)}) d\sigma(\lambda) = \int (x_x^{\varphi(\lambda)}, y_x^{*\varphi(\lambda)}) d\sigma(\lambda) = \int \pi_\lambda(x y_x) d\sigma(\lambda) = \int \pi_\lambda(x) d\sigma(\lambda)$.[3]

Remark. A semi-trace $\tau(\cdot)$ on a D^*-algebra $\mathfrak{A}$ is called pure, if π is not positively linear combination of any linearly independent semi-traces. Then π is pure if and only if $W^a \frown W^b = \{\lambda I\}$ where W^a and W^b are W^*-algebras generated by $\{x^a\}$ and $\{x^b\}$ in the corresponding two-sided representation $\{x^a, x^b, j, \mathfrak{H}\}$. (Cf. Prop. 2 of [3].) The Theorem 4 in the previous paper [3] follows as a special case of Th. in this paper, i.e. the case of the motion G containing only the identity automorphism: for any semi-trace τ of $\mathfrak{A}$ there exists a system of pure semi-traces π_λ such that $\tau(x) = \int \pi_\lambda(x) d\sigma(\lambda)$ for all $x \varepsilon \mathfrak{A}$ where $\sigma(\lambda)$ is similar with Th. (The proof of Th. 4 in the paper [3] has been remained as incomplete on choosing the $\sigma(\lambda)$-null set N such that π_λ are semi-traces for $\lambda \bar{\varepsilon} N$, cf. foot-note 11) of [3].)

References

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3) Let $\pi_0(\cdot)$ be arbitrary but fixed G-ergodic semi-trace. If we put $\pi_\lambda(x) = \pi_0(x)$ for all $\lambda \varepsilon N$ and $x \varepsilon \mathfrak{A}$, $\pi_\lambda(x)$ are determined for all $\lambda \varepsilon R$ and G-ergodic. Since N is $\sigma(\lambda)$-null set, the $\sigma(\lambda)$-integration of $\pi_\lambda(x) (x \varepsilon \mathfrak{A})$ over R is $\tau(x)$.