173. *Dirichlet Problem on Riemann Surfaces. II*

(*Harmonic Measures of the Set of Accessible Boundary Points*)

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(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1954)

Let $\underline{R}$ be a null-boundary Riemann surface with A-topology[1] and let R be a positive boundary Riemann surface given as a covering surface over $\underline{R}$. When a curve L on R converges to the boundary of R and its projection $\underline{L}$ on $\underline{R}$ tends to a point of $\underline{R}^*$, we say that L determines an accessible boundary point (A.B.P.) relative to $\underline{R}^*$. In the following we denote the set of all A.B.P.'s by $\mathfrak{A}(R, \underline{R}^*)$. We consider continuous super-harmonic function $v(z)$ in R such that $0\leqq v(z)\leqq 1$ and $\lim v(z)=1$ when z tends to the boundary along every curve determining an A.B.P. and we denote by $\mu(R, \mathfrak{A}(R, \underline{R}^*))$ the lower envelope of above functions which is harmonic in R on account of Perron-Brelot's theorem. We also consider $\mathfrak{A}(R^\infty, \underline{R}^*)$ and $\mu(R^\infty, \mathfrak{A}(R^\infty, \underline{R}^*))$ defined similarly on R^∞. In the following we assume that the universal covering surface of the projection of R on $\underline{R}$ is hyperbolic. Then there exists a null-boundary Riemann surface $\underline{R}'$ such that the projection of $R\subset\underline{R}'$, $\underline{R}'\subset\underline{R}$ and that $\underline{R}'^\infty$ is hyperbolic. We map $\underline{R}'^\infty$ and R^∞ conformally onto $U_\eta: |\eta|<1$ and $U_\xi: |\xi|<1$ respectively. Let l_ξ be a curve in U_ξ determining an A.B.P. of R^∞, whose projection on $\underline{R}'$. Then we see that l_ξ converges to a point $\xi_0: |\xi_0|=1$ and $\underline{z}=\underline{z}(\xi): U_\xi\rightarrow R\rightarrow\underline{R}'$ has an angular limit at ξ_0. It follows that $\underline{z}=\underline{z}(\xi)$ has angular limits at every point of A_ξ^l with respect to $\underline{R}'$, where A_ξ^l is the set of points ξ' on $|\xi|=1$ such that at least one curve determining A.B.P. with projection in $\underline{R}$ terminates at ξ'.

Let $\{R'_\lambda\}$ be an exhaustion of $\underline{R}'$ and $\Delta_{l,m,n}(\theta)$ be the set such that $\frac{1}{n}\leqq|\xi-e^{i\theta}|<\frac{1}{m}$ and $|\arg(1-e^{-i\theta}\xi)|<\frac{\pi}{2}-\frac{1}{l}$ and let $\delta(f(\xi))$ be the diameter of the set $f(\xi): \xi\in\Delta_{l,m,n}(\theta)$ with respect to the A-topology. Then we have

$$A_\xi^l=\mathcal{E}_{\theta}\Big[\sum_{\lambda}\prod_{l}\prod_{k}\sum_{m}\prod_{n}\delta(f(\xi))\leqq\frac{1}{k}\leftarrow\xi\in\Delta_{l,m,m+n}(\theta)\Big].$$

Since $\delta(f(\xi))$ is continuous with respect to θ for fixed l, m and n, this shows that A_ξ^l is a Borel set.

M. Ohtsuka has proved the next

1) See, Dirichlet problem. I.

Theorem 2.1. If the universal covering surface of the projection is of hyperbolic type, we have

$$\omega(U_\xi, A'_\xi)=\mu(R^\infty, \mathfrak{A}(R^\infty, \underline{R}^*)),^{2)}$$

where $\omega(U_\xi, A'_\xi)$ *is the harmonic measure of* A'_ξ.

Theorem 2.2. Let the universal covering surface of the projection of R *be hyperbolic and map* R^∞ *onto* $U_\xi: |\xi|<1$ *conformally. Let* D *be the normal polygon of Fuchsian group containing* $\xi=0$ *with arcs* α_i $(i=1, 2, \ldots)$ *on* $|\xi|=1$ *and let* T_j $(j=1, 2, \ldots)$ *be the substitutions of Fuchsian group.*

If mes $(\sum_j T_j(\sum_i \alpha_i))=2\pi$, *we have*

$$\mu(R^\infty, \mathfrak{A}(R^\infty, \underline{R}^*))=\mu(R, \mathfrak{A}(R, \underline{R}^*)).$$

Proof. Let A_ξ be the set of points such that at least one curve determining an A.B.P. on $\mathfrak{A}(R, \underline{R}^*))$ terminates there. Then A_ξ is measurable and $\mu(R^\infty, \mathfrak{A}(R^\infty, \underline{R}^*))\overset{3)}{=}\mu(R^\infty, \mathfrak{A}(R^\infty, \underline{R}))=\omega(U_\xi, A_\xi)$. Assume $\mu(R, \mathfrak{A}(R, \underline{R}^*))\gneqq\mu(R^\infty, \mathfrak{A}(R^\infty, \underline{R}^*))$, then there exists, out side of A_ξ, a set E_δ of positive measure such that $\mu(R, \mathfrak{A}(R, \underline{R}^*))$ has angular limits larger than δ $(\delta>0)$.

Since $\mathrm{mes}(E_\delta\cap(\sum_j T_j\sum_i \alpha_i))=\mathrm{mes}\, E_\delta$, there exist $\{\alpha_i\}$ and an integer m such that $\mathrm{mes}\,(\sum_j T_j(\sum_{i=1}^m(\alpha_i-\alpha'_i))<\frac{1}{4}\mathrm{mes}\, E_\delta$, where α'_i is a sub-arc of α_i such that α_i and α'_i have no common endpoints. Take closed sets F^i_δ $(i=1, 2, \ldots m)$ such that $F^i_\delta\subset(\alpha'_i\cap E_\delta)$ and $\mathrm{mes}\,(\sum_j T_j(\sum_{i=1}^m F^i_\delta))>\frac{1}{2}\mathrm{mes}\, E_\delta$. Denote by $\omega(F)$ the harmonic measure of $\sum_j T_j(\sum_{i=1}^m F^i_\delta)$. Then it is automorphic with respect to Fuchsian group and $\omega(F)$ has limit zero along every curve terminating at $(\alpha_i\cap$ complement of $\sum_{i=1}^m F^i_\delta)$. Put $J^\lambda_l=\mathop{\mathfrak{E}}_z[\omega(F)\geqq\lambda]\cap C_l(\sum_{i=1}^m F^i_\delta)$, where $C_l(\sum_{i=1}F^i_\delta)$ is the set of point z of R such that $\mathrm{dist}\,(z, \sum^m F^i_\delta)>\frac{1}{l}$ on D, and let $\omega^{\lambda,l}_{m,m+i}(z)$ be a harmonic function in $R_{m+i}-\{(R_{m+i}-R_m)\cap J^\lambda_l\}$ such that $\omega^{\lambda,l}_{m,m+i}(z)=0$ on $\partial R_{m+i}-J^\lambda_l$ and $\omega^{\lambda,l}_{m,m+i}(z)=1$ on $\partial J^\lambda_l\cap(R_{m+i}-R_m)+\partial R_m\cap J^\lambda_l$. Then it is clear $\omega^{\lambda,l}_m(z)=\lim_{i=\infty}\omega^{\lambda,l}_{m,m+i}(z)$ is super-harmonic and $\omega^{\lambda,l}(z)=\lim_{m=\infty}\omega^{\lambda,l}_m(z)$ has limit 0 on $\alpha_i\cap F^i_\delta$ and moreover $\omega(F)-\lambda\omega^{\lambda,l}(z)\geqq 0$. Thus $\omega^{\lambda,l}(z)=0$, because $\omega^{\lambda,l}(z)$ has the angular limit 0 almost everywhere on $|\xi|=1$. Hence we can easily construct a super-harmonic function $w(z)$ such that $w(z)=\infty$ along every curve tending to the

2) This theorem is proved under a little weaker condition. See, M. Ohtsuka: On covering surfaces over an abstract Riemann surface, Nagoya Math. Journ., **4**, 109–118 (1952).

3) See 5).

boundary in $\sum_{l,\lambda>0}^{l=\infty} J_l^\lambda$. Then $S(z)=\mathrm{Min}\,[1,\ v(z)-\delta\omega(F)+\varepsilon w(z)]$ has the limit l along every curve determining an A.B.P. Hence $\mu(R,\mathfrak{A}(R,\underline{R}^*))\leqq\mu(R,\mathfrak{A}(R,\underline{R}^*))-\delta\omega(F)$. This is absurd, therefore

$$\mu(R,\mathfrak{A}(R^\infty,\underline{R}^*))\leqq\mu(R^\infty,\mathfrak{A}(R^\infty,\underline{R}^*)).$$

On the other hand $\mu(R^\infty,\mathfrak{A}(R^\infty,\underline{R}^*))\leqq\mu(R,\mathfrak{A}(R,\underline{R}^*))$, since every $v(z)$ on R can be considered on R^∞.

Corollary. If R is a Riemann surface of finite connectivity and R is hyperbolic, $\mu(R^\infty,\mathfrak{A}(R^\infty,\underline{R}^*))=\mu(R,\mathfrak{A}(R,\underline{R}^*))$.

Since R is a metric space, $R+\mathfrak{A}(R,\underline{R}^*)$ is also a metric space. Let l_ξ be a continuous curve in U_ξ such that whose projection L on R converges to a point of $\mathfrak{A}(R,\underline{R}^*)$ with respect to the metric of R. Then the projection $\underline{L}$ of L on $\underline{R}$ converges to a point $\underline{p}_0\in\underline{R}^*$. If $\underline{p}_0\in\underline{R}$, the composed function $\underline{z}=\underline{z}(\xi):R^\infty\to\underline{R}$ has an angular limit $\underline{p}_0$. It follows that l_ξ tends to a point ξ_0. Therefore $\underline{z}(\xi)$ has the angular limit $\underline{p}_0$. Hence $z(\xi):R^\infty\to R+\mathfrak{A}(R,\underline{R}^*)$ has the angular limit p_0. We denote by A_ξ'' the set on $|\xi|=1$ such that at least one l_ξ above-mentioned terminates. Let E_g be a set on $|\xi|=1$ where the Green's function of R has an angular limit 0. If l_ξ is a Stolz's path terminating at $\xi_0\in A_\xi'\cap E_g$, then the projection L of l_ξ on R tends to the boundary of R and has the projection $\underline{L}$ on $\underline{R}$ which tends to a point $\underline{p}\in\underline{R}$. Thus l_ξ determines an A.B.P. of $\mathfrak{A}(R,\underline{R})$. Hence

$$A_\xi\overset{4)}{\supset}A_\xi''\supset(A_\xi'\supset E_g).$$

Let $\mathfrak{F}$ be a closed subset of $\mathfrak{A}(R,\underline{R}^*)$ and let F be the set on $|\xi|=1$ such that at least one curve determining an A.B.P. of $\mathfrak{F}$ terminates. We call F the hyper image ot $\mathfrak{F}$. Put

$$F'=\underset{\theta}{\mathcal{E}}\left[\prod_l\prod_k\sum_m\prod_n(\text{distance }(f(r_{m,m+n}(\theta)).\ \mathfrak{F})\leqq\frac{1}{k}\right],$$

where $r_{m,m+n}(\theta)$ is a segment of the radius such that $1-\dfrac{1}{m}\leqq|\xi|<1-\dfrac{1}{n}$, $\arg\xi=\theta$. Hence F' is measurable. We call F' the image of $\mathfrak{F}$. Then

$$(F'\cap A_\xi')\subset F\subset\{B_\xi+(A_\xi'\cap F')\}\quad\text{and}$$

$$(F\cap A_\xi'\cap E_g)\subset F''\subset F,$$

where B_ξ is the set where at least one curve determining an A.B.P. of R whose projection lies on B of $\underline{R}$ terminates.

Let $\mu(R^\infty,\mathfrak{F})$ be the lower envelope of super-harmonic functions $v(z)$ such that $0\leqq v(z)\leqq1$, $\lim v(z)=1$ along every curve determining an A.B.P. of R^∞ lying on $\mathfrak{F}$ and let $\mu(R^\infty,\mathfrak{A}(R^\infty,B))$, $\mu(\underline{R}^\infty,\mathfrak{A}(\underline{R}^\infty,B))$

4) See, the proof of theorem 2.2. $\cdots A_\xi$ is a hyper image of $\mathfrak{A}(R^\infty,\underline{R}^*)$.

be the lower envelopes of $v_B(R^\infty)$ and $v_B(\underline{R}^\infty)$ such that $\lim v_B(R^\infty)=1$, $\lim v_B(\underline{R}^\infty)=1$ along every curve determining an A.B.P. of R^∞ and $\underline{R}^\infty$ lying on B of $\underline{R}$. Since at every point of $F'\cap A'_\xi$, $\lim v(z)=1$ along radial segments, $\omega(U_\xi, F'\cap A_\xi)\leqq\mu(R^\infty, F\cap A_\xi)\leqq\mu(R^\infty,\mathfrak{F})\leqq\mu(R, \mathfrak{A}(R,B)+\mu(R^\infty, A'_\xi\cap F')$. Since $\mu(R^\infty,\mathfrak{A}(R^\infty,B))\leqq\mu(\underline{R}^\infty,\mathfrak{A}(\underline{R}^\infty,B))=0$,[5] $\mathrm{mes}\,|A_\xi-A''_\xi|=0$ and since $\omega(U_\xi,F'\cap A_\xi)\geqq\mu(R^\infty, A_\xi\cap F'')$, we have $\omega(U_\xi,F')=\omega(U_\xi,F'\cap A_\xi)=\mu(R^\infty,\mathfrak{F})=\omega(U_\xi,F'')$, because $\mathrm{mes}\,|E_g|=2\pi$.

Theorem 2.3. Let R be a positive boundary Riemann surface and let the universal covering surface of the projection of R over $\underline{R}$ be hyperbolic, if $\mu(R,\mathfrak{A}(R,\underline{R}^*))=\mu(R^\infty,\mathfrak{A}(R^\infty,\underline{R}^*))$, then

$$\mu(R,\mathfrak{F})=\mu(R^\infty,\mathfrak{F})=\omega(U_\xi,F')$$

for every closed subset $\mathfrak{F}$ of $\mathfrak{A}(R,\underline{R}^*)$.

Proof. Put $\mathfrak{A}(R,\underline{R}^*)$ in the place of $\mathfrak{F}$ in the above equality and regard that $z=z(\xi)$ has angular limits on $\underline{R}$ at a set on $|\xi|=1$ where at least one curve determining an A.B.P. of $\mathfrak{A}(R,\underline{R})$ terminates. Then we have $\mu(R^\infty,\mathfrak{A}(R,\underline{R}^*))\,\mu(R^\infty,\mathfrak{A}(R,\underline{R}))=\omega(U_\xi,A_\xi)$. We denote by $\mathfrak{F}_n$ all points z of $R+\mathfrak{A}(R,\underline{R}^*)$ such that z has a distance $\leqq\frac{1}{n}$ from $\mathfrak{F}$. Then $\mathfrak{F}=\bigcap_n\mathfrak{F}_n$, and the image F_n of $\mathfrak{F}_n$ on $|\xi|=1$ is measurable and $z=z(\xi)$ has angular limits on R at F'_n. Let $\{R_m\}$ be an exhaustion of R with compact relative boundaries $\{\partial R_m\}$ and let $\partial\mathfrak{F}_n$ be the relative boundary of $\mathfrak{F}_n$. Let $\omega^n_{m,m+i}(z)\,(n,m,i=1,2,\dots)$ be the harmonic function in $R_{m+i}-(F_n\cap(R_{m+i}-R_m))$ such that $\omega^n_{m,m+i}(z)=1$ on $\{\partial\mathfrak{F}_n\cap(R_{m+i}-R_m)\}+(\mathfrak{F}_n\cap\partial R_m)$ and $\omega^n_{m,m+i}(z)=0$ on $\partial R_{m+i}-\mathfrak{F}_n$. Then $\omega^n_m(z)=\lim_{i=\infty}\omega^n_{m,m+i}(z)$ is super-harmonic in R and $\lim_m\omega^n_m(z)\geqq\mu(R,\mathfrak{F}_n)$ for every n. Assume, $\mu(R,\mathfrak{F})\gneqq\mu(R^\infty,\mathfrak{F})$. Then there exist a number n and a closed subset E_δ in $A'_\xi\cap CF'_n$ for sufficiently small number δ such that $\mu(R,\mathfrak{F}_n)$ has angular limits larger than δ and $z=z(\xi)$ converges uniformly inside an angular domain: $|\arg|\xi-e^{i\theta}||<\frac{\pi}{2}-\delta'$ $(\delta'>0)$ for every point $e^{i\theta}$ of E_δ, because $\omega(U_\xi,A'_\xi)=\mu(R,\mathfrak{A}(R,\underline{R}^*))\geqq\mu(R,\mathfrak{F})$ implies that $\mu(R,\mathfrak{F})$ has angular limits 0 almost everywhere on CA_ξ (complementary set of A_ξ). Let $D_\lambda(E_\delta)$ be the domain in U_ξ such that $D_\lambda(E_\delta)$ contains the endpart of the angular domain $|\arg(1-e^{-i\theta}\xi)|<\frac{\pi}{2}-\lambda$ at every point $e^{i\theta}$ of E_δ. On the other hand the universal covering surface R^∞_m is mapped onto a simply connected domain containing $\xi=0$ such that $\bigcup_m R^\infty_m=U_\xi$. Let H_r be the ring

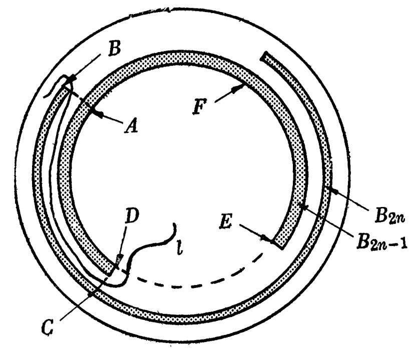
5) Map R of a null-boundary Riemann surface onto $U_\xi:|\xi|<1$ and let E the image of the ideal boundary of R. Then $\mathrm{mes}\,E=0$. See, M. Tuji: Some metrical theorems on Fuchsian groups, Kodai Math. Sem. Rep., Nos. 4-5, 27-44 (1950).

domain such that $r<|\xi|<1$. Then there exists r such that $z=z(\xi)$ has a distance $\geqq\frac{1}{2n}$ from $\mathfrak{F}_n$, where $\xi\in H_r\cap D_\lambda(E_\delta)$, because $z=z(\xi)$ has an angular limits of A.B.P.'s of R which have a distance $\geqq\frac{1}{n}$ from $\mathfrak{F}$. Let $D'_\lambda(E_\delta)$ be a component of $H_r\cap D_\lambda(E_\delta)$ which has a closed subset of positive measure of E_δ and let $\omega(\xi)$ be a harmonic function such that $\omega(\xi)=1$ on the boundary of $D'_\lambda(E_\delta)$ except one on $|\xi|=1$ and $\omega(\xi)=0$ on $|\xi|=1$. Consider $\omega^n_m(z)$ in U_ξ, we see easily that $\omega^{2n}_m(z)\leqq\omega(\xi)$ for every m, because the image of $\partial\mathfrak{F}_n$ does never fall in $D'_\lambda(E_\delta)$. Since the boundary of $D'_\lambda(E_\delta)$ is rectifiable, there exists a set of positive measure on $|\xi|=1$ where $\omega(\xi)=0$. Hence $\mu(R,\mathfrak{F})\leqq\lim_n\omega^{2n}(z)\leqq\omega(\xi)$, whence $\mu(R,\mathfrak{F})$ has an angular limit 0 almost everywhere on E_δ. This is a contradiction. Thus we have

$$\mu(R,\mathfrak{F})=\mu(R^\infty,\mathfrak{F}).$$

Let $\underline{R}'$ be the projection of R on $\underline{R}$. If $\underline{R}'^\infty$ of R is parabolic ($\underline{R}'^\infty$ cannot be mapped onto a unit-circle conformally) we remove a finite number of points $p_1, p_2, \dots, p_n$ (if R is closed and its genus is zero or one, then the number of points which are to be remove, is three or one respectively) and remove all the points p_{ij} $(j=1,2,\dots)$ lying on p_i from R. Denote the remaining surface by $\widetilde{R}$ and define $\mu(\widetilde{R},\mathfrak{A}(\widetilde{R},\underline{R}^*))$ and $\mu(\widetilde{R}^\infty,\mathfrak{A}(\widetilde{R}^\infty,\underline{R}^*))$ similarly. In the following we assume that R has at least one A.B.P. M. Ohtsuka has proved the following:

$\mu(R,\mathfrak{A}(R,\underline{R}^*))=\mu(\widetilde{R},\mathfrak{A}(\widetilde{R},\underline{R}^*))\geqq\mu(R^\infty,\mathfrak{A}(R^\infty,\underline{R}^*))\geqq\mu(\widetilde{R}^\infty,\mathfrak{A}(\widetilde{R}^\infty,R^*))$ and if R is a null-boundary Riemann surface, $\mu(R,\mathfrak{A}(R,\underline{R}^*))=\mu(\widetilde{R},\mathfrak{A}(\widetilde{R},\underline{R}^*))=1$ and $\mu(R^\infty,\mathfrak{A}(R^\infty,\underline{R}^*))=0$. He proposed the problem: does there exist a case when the inequality holds? We show that there are these cases.



Example. Let $B_{2n}, B_{2n-1}: n=1,2,\dots$ be the system of closed domains in $|z|<1$ such that

$$B_{2n}:\ 1-\frac{1}{4n+3}\leqq r\leqq 1-\frac{1}{4n+4}:\ \frac{3}{4}\pi\leqq\theta\leqq\frac{\pi}{4}\quad\left(\text{containing } -\frac{\pi}{2}\right)$$

$$B_{2n-1}:\ 1-\frac{1}{4n}\leqq r\leqq 1-\frac{1}{4n+1}:\ -\frac{3}{4}\pi\geqq\theta\geqq-\frac{5\pi}{4}\quad\left(\text{containing } \frac{\pi}{2}\right).$$

We can construct a holomorphic function $f(z):|z|<1$ by Runge's theorem such that $|f(z)-1|<\frac{1}{n}$ in B_{2n} and $|f(z)|<\frac{1}{n}$ in

B_{2-1}. It is clear that $f(z)$ is not bounded in $|z|<1$. Since the value $w=f(z)=\infty$ is an exceptional point, there exists at least one asymptotic path along which $f(z)$ tends to ∞, when z converges to the boundary of the unit-circle. Let l be an asymptotic path with starting point p_0 where $|f(p_0)|=M_0$. Then l is not contained in $\sum_{m=n_0}^{\infty}(B_{2n}+B_{2n-1})$ for a number n_0 and determines an A.B.P. $\mathfrak{P}$ lying over $w=\infty$. Let p_r be the first point where l passes $|z|=r$, and let l_r be the part of l between p_0 and p_r. Let $v_l(z)$ be a continuous super-harmonic function such that $0\leqq v_l(z)\leqq 1$ and $\lim v_l(z)=1$, when z tends to $|z|=1$ along l. Then there exists r_δ such that $v_l(z)\geqq 1-\delta$ on $l-l_{r_\delta}$ for a given number δ. Consider the part $l_{r_2r_1}=l_{r_2}-l_{r_1}$, then $l_{r_2r_1}$ connects two circles $|z|=r_1$ and $|z|=r_2$ $(r_2>r_1>1-\frac{1}{4n}:n\geqq n_0)$. Without loss of generality, we can suppose that l has a branch in the left semi-circle. Let A, B, C, D, E and F be points shown in the figure and let D_n be the simply connected domain with boundary $\overline{AB}+\widehat{BC}+\overline{CD}+\widehat{DEFA}$ and let $\omega_n(z)$ be the harmonic measure of $\widehat{BC}$ with respect to D_n. Then we see that $v_l(z)\geqq(1-\delta)\omega_n(z)$ and $v_l(0)\geqq(1-\delta)\omega_n(0)\geqq\delta_1$ $(\delta_1>0)$ for every n. We denote by U the unit-circle and let $v(U,z,\mathfrak{P})$ be a continuous super-harmonic function such that $0\leqq v(U,z,\mathfrak{P})\leqq 1$ and $\lim v(U,z,\mathfrak{P})=1$ along l every curve tending to $\mathfrak{P}$ and let $\mu(U,z,\mathfrak{P})$ be their lower envelope. Since $\{v(U,z,\mathfrak{P})\}$ is contained in the class $\{v_l(z)\}$, we have $\mu(U,z,\mathfrak{P})\geqq\delta_1$ at 0.

We remove all points $\{z_i\}$ where $f(z_i)=0$, or 1 or 2 from U_1 and denote by $\tilde{U}$ the remaining surface. Map $\tilde{U}^{\infty}$ onto U_ξ: $|\xi|<1$ conformally. Let $\{\mathfrak{P}_\infty\}$ be the set of all A.B.P.'s of $\tilde{U}$ whose projection lie on $w=\infty$ and let $E_{\xi,\infty}$ be the hyper image of $\{\mathfrak{P}_\infty\}$. Then $E_{\xi,\infty}$ is a set of linear measure zero. Let $v(U_\xi,\{\mathfrak{P}_\infty\})$ and $v(U_\xi,E_{\xi,\infty})$ be super-harmonic function in U_ξ such that $\lim v(U_\xi,\{\mathfrak{P}_\infty\})=1$ when z tends to $E_{\xi,\infty}$. Then

$\mu(\tilde{U}^{\infty},\mathfrak{P})\leqq\mu(\tilde{U}^{\infty},\{\mathfrak{P}_\infty\})\leqq\omega(U_\xi,E_{\xi,\infty})=0$, where $\mu(\tilde{U}^{\infty},\mathfrak{P})$ and $\mu(\tilde{U}^{\infty},\{\mathfrak{P}_\infty\})$ are the lower envelopes of $v(U_\xi,\mathfrak{P})$ and $v(U_\xi,\{\mathfrak{P}_\infty\})$.

Since $\mathfrak{P}$ is closed, we can conclude by Theorem 2.3 that

$$\mu(U,\mathfrak{P})=\mu(\hat{U},\mathfrak{P})\ \mu\gneqq(\tilde{U}^{\infty},\mathfrak{P})\ \text{implies}$$

$$\mu(\tilde{U},\mathfrak{A}(U,\underline{R}^*))\gneqq\mu(\tilde{U}^{\infty},\mathfrak{A}(U,\underline{R}^*)).$$

We consider U as a Riemann surface R, then we have

$$\mu(R^\infty,\mathfrak{A}(R^\infty,\underline{R}^*))\gneqq\mu(\tilde{R}^{\infty},\mathfrak{A}(\tilde{R}^{\infty},\underline{R}^*)).$$

Similarly, if we consider $\tilde{U}$ as a Riemann surface R, then we have

$$\mu(R,\mathfrak{A}(R,\underline{R}^*))\gneqq\mu(R^\infty,\mathfrak{A}(R^\infty,\underline{R}^*)).$$