168. *On the Strong Summability of the Derived Fourier Series*

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1. Let $f(t)$ be a periodic function of bounded variation with period 2π, and its Fourier series be

$$a_0/2+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\sum_{n=0}^{\infty}A_n(t).$$

We shall consider the derived Fourier series

$$\sum_{n=1}^{\infty}n(b_n\cos nt-a_n\sin nt)=\sum_{n=1}^{\infty}A_n'(t)$$

and its conjugate series

$$\sum_{n=1}^{\infty}n(a_n\cos nt+b_n\sin nt)=\sum_{n=1}^{\infty}B_n'(t).$$

We denote by $\tau_n(t)$ and $\bar{\tau}_n(t)$ the nth partial sums of them, i.e.

$$\tau_n(t)=\sum_{m=1}^{n}m(b_m\cos mt-a_m\sin mt)=\sum_{m=1}^{n}A_m'(t),$$

$$\bar{\tau}_n(t)=\sum_{m=1}^{n}m(a_m\cos mt+b_m\sin mt)=\sum_{m=1}^{n}B_m'(t).$$

As in the case of Fourier series, we use the modified partial sums of them;

$$\tau_n^*(t)=\tau_n(t)-A_n'(t)/2,\quad \bar{\tau}_n^*(t)={}_n\bar{\tau}(t)-B_n'(t)/2.$$

Recently B. N. Prasad and U. N. Singh[1] proved the following theorems:

Theorem A. *If $f(t)$ is a continuous function of bounded variation which is differentiable at $t=x$ and if for some $\varepsilon>0$*

$$G(t)=\int_0^t|dg(u)|=o\left\{t\left(\log\frac{1}{t}\right)^{-1-\varepsilon}\right\},\ \text{as } t\to 0,$$

where $g(u)=g_x(u)=f(x+u)-f(x-u)-2uf'(x)$, then

$$\sum_{m=1}^{n}|\tau_m(x)-f'(x)|=o(n).$$

That is, the derived Fourier series of $f(t)$ is $(H, 1)$ summable to the sum $f'(x)$ at $t=x$.

Theorem B. *If $f(t)$ is a continuous function of bounded variation which is differentiable at $t=x$ and if for some $\varepsilon>0$*

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1) B. N. Prasad and U. N. Singh: Math. Zeits., **56**, 280–288 (1952).

$$H(t)=\int_0^t |dh(u)| = o\left\{t\left(\log\frac{1}{t}\right)^{-1-\varepsilon}\right\}, \text{ as } t\to 0,$$

where $h(u)=h_x(u)=f(x+u)+f(x-u)-2f(x)$, *then*

$$\sum_{m=1}^{n} |\bar{\tau}_m(x)-H_m(x)| = o(n),$$

where $H_n(x)=-\dfrac{1}{4\pi}\displaystyle\int_{1/n}^{\pi} h_x(t)\,\mathrm{cosec}^2\frac{t}{2}\,dt.$

In this paper we shall prove the following (H, k) summability theorems.

Theorem 1. *Under the assumption of Theorem A,*[2] *we have*

$$\sum_{m=1}^{n} |\tau_m^*(x)-f'(x)|^k = o(n), \text{ as } n\to\infty,$$

for any $k>0$.

Theorem 2. *Under the assumption of Theorem B, we have*

$$\sum_{m=1}^{n} |\bar{\tau}_m^*(x)-H_m(x)|^k = o(n), \text{ as } n\to\infty,$$

for any $k>0$.

2. Proof of Theorem 1.[3] We have

$$\tau_n^*(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{d}{dx}\left(\frac{\sin n(x-u)}{\tan(x-u)/2}\right)\right\}f(u)du$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\left(\frac{d}{dt}\frac{\sin nt}{\tan t/2}\right)dt = -\frac{1}{\pi}\int_0^{\pi}\{f(x+t)-f(x-t)\}\left(\frac{d}{dt}D_n^*(t)\right)dt,$$

where $D_n^*(t)=\dfrac{\sin nt}{2\tan t/2}$. Integrating by parts, we get

$$\tau_n^*(x)=\frac{1}{\pi}\int_0^{\pi} D_n^*(t)d\{f(x+t)-f(x-t)\}=\frac{1}{\pi}\int_0^{\pi} D_n^*(t)dg(t)+f'(x).$$

Thus we obtain

$$\tau_n^*(x)-f'(x)=\frac{1}{\pi}\int_0^{\pi} D_n^*(t)dg(t)=P_n,$$

say.

Hence, it is sufficient to show that

$$S_n^k=\sum_{m=1}^{n}|P_m|^k=o(n).$$

For this purpose we set $c_m=|P_m|^{k-1}\operatorname{sgn}P_m$, $\Lambda_n(t)=\sum_{m=1}^{n} c_m\sin mt$ and $\Gamma_n=\sum_{m=1}^{n}|c_m|$. Then $|\Lambda_n(t)|\leqq\Gamma_n$ and $|\Lambda_n(t)|\leqq nt\Gamma_n$. Using these formulas, we have

2) In our theorems, the continuity of the function $f(t)$ in the whole interval is not necessary.

3) *Cf.* G. H. Hardy and J. E. Littlewood: Proc. London Math. Soc., **26**, 273-286 (1926-27).

$$2\pi S_n^k = 2\pi \sum_{m=1}^{n} |P_m|^{k-1} P_m \operatorname{sgn} P_m = 2\pi \sum_{m=1}^{n} c_m P_m$$
$$= \sum_{m=1}^{n} c_m \int_0^{\pi} \cot \frac{t}{2} \sin mt \, dg(t) = \int_0^{\pi} \Lambda_n(t) \cot \frac{t}{2} dg(t)$$
$$= \int_0^{1/n} + \int_{1/n}^{\pi} = I_1 + I_2,$$

say, where

$$|I_1| \leqq n\Gamma_n \int_0^{1/n} |dg(t)| = o(\Gamma_n)$$

and

$$|I_2| \leqq \Gamma_n \int_{1/n}^{\pi} \cot \frac{t}{2} |dg(t)|$$
$$\leqq \Gamma_n \left\{ \left[\cot \frac{t}{2} G(t) \right]_{1/n}^{\pi} + \frac{1}{2} \int_{1/n}^{\pi} \operatorname{cosec}^2 \frac{t}{2} G(t) dt \right\}$$
$$= o(\Gamma_n) + o\left(\Gamma_n \int_{1/n}^{\pi} \operatorname{cosec}^2 \frac{t}{2} \frac{t dt}{(\log 1/t)^{1+\varepsilon}} \right)$$
$$= o(\Gamma_n) + o\left(\Gamma_n \int_{1/n}^{\pi} \frac{dt}{t(\log 1/t)^{1+\varepsilon}} \right) = o(\Gamma_n).$$

Thus we get $S_n^k = o(\Gamma_n)$. However, by Hölder's inequality, we can see,[4]

$$\Gamma_n \leqq \left(\sum_{m=1}^{n} |P_m|^k\right)^{1/k'} n^{1/k} = S_n^{k/k'} n^{1/k} \quad (1/k + 1/k' = 1).$$

Hence we have $S_n^k = o(S_n^{k/k'} n^{1/k})$, that is, $S_n^k = o(n)$.

3. Proof of Theorem 2. As usual we put

$$\bar{D}_n^*(t) = \frac{1 - \cos nt}{2 \tan t/2}, \text{ then}$$

$$\bar{\tau}_n^*(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \left(\frac{d}{dx} \bar{D}_n^*(u - x) \right) du$$
$$= -\frac{1}{\pi} \int_0^{\pi} \left(\frac{d}{dt} \bar{D}_n^*(t) \right) \{ f(x+t) + f(x-t) \} dt$$
$$= -\frac{1}{\pi} \int_0^{\pi} \bar{D}_n^*(t) dh(t) = -\frac{1}{\pi} \left(\int_0^{1/n} + \int_{1/n}^{\pi} \right) = J_1 + J_2,$$

say, where

$$|J_1| \leqq O\left(\int_0^{1/n} \frac{1}{t} (nt) |dh(t)| \right) = O\left(n \int_0^{1/n} |dh(t)| \right) = o(1)$$

4) We may suppose $k > 1$.

and

$$J_2=-\frac{1}{2\pi}\int_{1/n}^{\pi}\cot\frac{t}{2}dh(t)+\frac{1}{2\pi}\int_{1/n}^{\pi}\cot\frac{t}{2}\cos nt\,dh(t)$$

$$=-\frac{1}{2\pi}\left[\cot\frac{t}{2}h(t)\right]_{1/n}^{\pi}-\frac{1}{2\pi}\int_{1/n}^{\pi}\frac{1}{2}\operatorname{cosec}^2\frac{t}{2}h(t)dt$$

$$+\frac{1}{2\pi}\int_{1/n}^{\pi}\cot\frac{t}{2}\cos nt\,dh(t)$$

$$=-\frac{1}{2\pi}\cot\frac{1}{2n}h\left(\frac{1}{n}\right)+H_n+T_n=o(1)+H_n+T_n.$$

Thus we get

$$\bar{\tau}_n^*-H_n=T_n+o(1).$$

Hence, it suffices to show that

$$\bar{S}_u^k=\sum_{m=1}^{n}|T_m|^k=o(n).$$

Similarly as in the proof of Theorem 1, we put $\bar{c}_m=|T_m|^{k-1}\operatorname{sgn}T_m$, $\bar{\Lambda}_n(t)=\sum_{m=1}^{n}\bar{c}_m\cos mt$ and $\bar{\Gamma}_n=\sum_{m=1}^{n}|\bar{c}_m|$. Then we have

$$\bar{S}_n^k=\int_{1/n}^{\pi}\bar{\Lambda}_n(t)\cot\frac{t}{2}dh(t)\leqq\bar{\Gamma}_n\int_{1/n}^{\pi}\cot\frac{t}{2}|dh(t)|$$

$$=o(\bar{\Gamma}_n)+o\left(\bar{\Gamma}_n\int_{1/n}^{\pi}\frac{dt}{t(\log 1/t)^{1+\varepsilon}}\right)=o(\bar{\Gamma}_n)=o(\bar{S}_n^{k/k'}n^{1/k}).$$

Thus we get $\bar{S}_n^k=o(n)$, which is required.

4. Finally we shall state the following, which may be similarly proved as Theorems 1 and 2.

Theorem 3. Under the assumption of Theorem A, we have

$$\sum_{m=1}^{n}|\tau_m(x)-f'(x)|^k=O(n),\ \text{for any } k>0.$$

Theorem 4. Under the assumption of Theorem B, we have

$$\sum_{m=1}^{n}|\bar{\tau}_m(x)-H_m(x)|^k=O(n),\ \text{for any } k>0.$$