28. On the Riesz Logarithmic Summability of the Conjugate Derived Fourier Series. I

By Masakiti KINUKAWA
Mathematical Institute, Tokyo Metropolitan University, Tokyo

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1. Let $f(x)$ be an integrable function with period 2π and its Fourier series be

(1.1) $$f(x)\sim a_0/2+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\equiv\sum_{n=0}^{\infty}A_n(x).$$

We call the series

(1.2) $$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)\equiv\sum_{n=1}^{\infty}B_n(t),$$

$$\sum_{n=1}^{\infty}n(b_n\cos nx-a_n\sin nx)=\sum_{n=1}^{\infty}A_n'(t)$$

and

(1.3) $$\sum_{n=1}^{\infty}n(a_n\cos nx+b_n\sin nx)=\sum_{n=1}^{\infty}nA_n(x)$$

conjugate series, derived series and conjugate derived series of (1.1), respectively.

The infinite series $\sum a_n$ is said to be summable by Riesz's logarithmic mean of order α, or simply summable (R, log, α), to sum s, provided that

$$R_\alpha(\omega)=\frac{1}{(\log\omega)^\alpha}\sum_{n<\omega}(\log\omega/n)^\alpha a_n$$

tends to a limit s, as $\omega\to\infty$.

The summability by Riesz's logarithmic means of the Fourier series was treated by Hardy [1], Takahashi [3], and Wang [4], [5], [6]. Wang has proved the Riesz summability analogue of Bosanquet's theorem concerning Cesàro summability of Fourier series. This theorem was extended to the derived Fourier series by Matsuyama [2]. In this paper we shall prove the analogue for the conjugate derived Fourier series and some related theorems.

We shall introduce some notations. Let us put

$$g_0(t)=g(t),$$

$$g_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_t^\pi\left(\log\frac{u}{t}\right)^{\alpha-1}\frac{g(u)}{u}\,du\qquad(\alpha>0).$$

Then $g_\alpha(t)\Big/\left(\log\frac{1}{t}\right)^\alpha$ is called the Riesz logarithmic mean of $g(t)$ of order α. If the Riesz logarithmic mean of $g(t)-s$ tends to zero as $t\to 0$, then we write

$$\lim_{t\to 0}g(t)=s\ \ (R, log, \alpha).$$

We denote by $g_\alpha^\beta(t)$ the βth integral of $g_\alpha(t)$, that is,

$$g_\alpha^\beta(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-u)^{\beta-1}g_\alpha(u)du,$$

$$g_\alpha^0(t)=g_\alpha(t).$$

2. In what follows we put

$$\varphi(t)=\varphi(x,t)=\frac{1}{2}\left\{f(x+t)+f(x-t)-2s\cos t\right\},$$

$$g(t)=\int_t^\infty \frac{\varphi(u)}{u^2}du$$

and suppose that $\varphi(t)/t$ is integrable in $(0,\infty)$. Then our theorems are stated as follows.

Theorem 1. *If*

$$\lim_{t\to 0} g(t)=0 \qquad (R, log, \alpha),$$

then the conjugate derived Fourier series of $f(t)$ is summable $(R, log, \alpha+2)$ to s at the point x, where $\alpha\geqq 0$.

Theorem 2. *If we suppose*

$$\int_0^t g_\alpha(u)du=g_\alpha^1(t)=o\left[t\left(\log\frac{1}{t}\right)^\alpha\right]$$

and

$$\int_t^\pi \frac{|g_\alpha(u+t)-g_\alpha(u)|}{u}du=o\left[\left(\log\frac{1}{t}\right)^{\alpha+2}\right],$$

then the conjugate derived Fourier series of $f(t)$ is summable $(R, log, \alpha+2)$ to s at the point x, where $\alpha\geqq 0$.

Theorem 3. *If the conjugate derived Fourier series is summable (R, log, α), then we have*

$$\lim_{t\to 0} g(t)=0 \qquad (R, log, \alpha+1+\varepsilon),$$

where $\alpha\geqq 2$ and ε is a positive number.

3. We start by some lemmas which need for the proof of our theorems.[1]

Lemma 1. *Let us put*

$$S_\alpha(t)=\int_0^1\left(\log\frac{1}{u}\right)^\alpha \sin ut\, du$$

for $\alpha>-1$. Then we have the following relations:

$$S_\alpha(t)=\begin{cases} O(1) & \text{for } t>0 \text{ and } \alpha>-1,\\ O[(\log t)^\alpha/t] & \text{for } t\geqq 2,\ \alpha\geqq 0,\\ O[(\log t)^{\alpha-1}/t] & \text{for } t\geqq 2,\ 0<\alpha<1,\end{cases} \tag{3.1}$$

$$S'_\alpha(t)=\begin{cases} O[(\log t)^\alpha/t^2] & \text{for } t\geqq 2,\ \alpha\geqq 1,\\ O(1/t^{1+\alpha}) & \text{for } t\geqq 2,\ 0\leqq\alpha<1,\end{cases} \tag{3.2}$$

$$S''_\alpha(t)=O[(\log t)^\alpha/t^3] \qquad \text{for } t\geqq 2,\ \alpha\geqq 1, \tag{3.3}$$

$$S_\alpha(0)=0 \quad (\alpha>-1), \tag{3.4}$$

1) *Cf.* Matsuyama [2] and Wang [4], [6], [7].

(3.5) $$S_0(t)=(1-\cos t)/t,$$

(3.6) $$[tS_\alpha(t)]'=\alpha S_{\alpha-1}(t) \qquad for\ \alpha>0,$$

(3.7) $$S_{r+s+1}(t)=\frac{\Gamma(r+s+2)}{\Gamma(r+1)\,\Gamma(s+1)}\int_0^1 S_s(ut)\left(\log\frac{1}{u}\right)^r du$$
$$for\ r>-1,\ s>-1.$$

Lemma 2.
$$\frac{2}{\pi}\int_0^\infty S_\alpha(u)\sin xu\,du=\left(\log\frac{1}{x}\right)^\alpha \qquad for\ 0<x<1,$$
$$=0 \qquad for\ x\geqq 1.$$

4. Proof of Theorem 1. The (R, log, β) means of the conjugate derived Fourier series is denoted by
$$R_\beta(\omega)=\frac{1}{(\log\omega)^\beta}\sum_{n<\omega}\left(\log\frac{\omega}{n}\right)^\beta nA_n(x),$$
and the Fourier series of $\varphi(t)$ becomes
$$\varphi(t)\sim\sum_{n=0}^\infty A_n(x)\cos nt-s\cos t.$$
Since $S'_\beta(t)$ and $S''_\beta(t)$ $(\beta\geqq 1)$ are integrable in $(0,\infty)$, by Young's theorem, we get
$$\int_0^\infty S'_\beta(\omega t)\varphi(t)dt=\sum_{n=1}^\infty A_n\int_0^\infty S'_\beta(\omega t)\cos nt\,dt-s\int_0^\infty S'_\beta(\omega t)\cos t\,dt,$$
where the nth term of the right side series is
$$\int_0^\infty S'_\beta(\omega t)\cos nt\,dt=\left[\frac{1}{\omega}S(\omega t)\cos nt\right]_0^\infty+\frac{n}{\omega}\int_0^\infty S_\beta(\omega t)\sin nt\,dt$$
$$=\begin{cases}\dfrac{\pi}{2}\dfrac{n}{\omega^2}\left(\log\dfrac{\omega}{n}\right)^\beta & \text{for } \dfrac{n}{\omega}<1,\\ 0 & \text{for } \dfrac{n}{\omega}\geqq 1,\end{cases}$$
by Lemma 2. Hence
$$\frac{2}{\pi}\int_0^\infty S'_\beta(\omega t)\varphi(t)\,dt=-\frac{1}{\omega^2}\left(\log\frac{\omega}{1}\right)^\beta s+\sum_{n<\omega}A_n(x)n\left(\log\frac{\omega}{n}\right)^\beta\frac{1}{\omega^2}.$$
Therefore
$$R_\beta(\omega)-s=\frac{2}{\pi}\frac{\omega^2}{(\log\omega)^\beta}\int_0^\infty S'_\beta(\omega t)\varphi(t)dt$$

(4.1) $$=\frac{2}{\pi}\frac{\omega}{(\log\omega)^\beta}\int_0^\infty\frac{\varphi(t)}{t}\Big[\beta S_{\beta-1}(\omega t)-S_\beta(\omega t)\Big]dt,$$

by Lemma 1, (3.6). If we put $\varphi(t)/t=\xi(t)$, then, by integration by parts, we get
$$\frac{\omega}{(\log\omega)^\beta}\int_p^\infty \xi(t)S_{\beta-1}(\omega t)\,dt$$
$$=\frac{\omega}{(\log\omega)^\beta}\Big[\xi^1(t)S_{\beta-1}(\omega t)\Big]_p^\infty-\frac{\omega^2}{(\log\omega)^\beta}\int_p^\infty\xi^1(t)S'_{\beta-1}(\omega t)dt$$
$$=R_1+R_2,$$

where p is a sufficiently large but fixed number. Assuming $\beta \geqq 2$, and by (3.2) and $\int_0^t \varphi(u)/u\, du = O(1)$, we get

$$R_1 = \frac{\omega}{(\log \omega)^\beta}\left[O\left\{\frac{(\log \omega t)^{\beta-1}}{\omega t}\right\}\right]_p^\infty = O\left(\frac{1}{p}\frac{1}{\log \omega}\right)$$

and

$$R_2 = O\left\{\frac{\omega^2}{(\log \omega)^\beta}\int_p^\infty \frac{(\log \omega t)^{\beta-1}}{\omega^2 t^2}dt\right\} = O\left(\frac{1}{p}\frac{1}{\log \omega}\right).$$

Similarly we get

$$\frac{\omega}{(\log \omega)^\beta}\int_p^\infty \xi(t)S_\beta(\omega t)\, dt = O(1/p).$$

On the other hand we have, by integration by parts and by (3.1), (3.6),

$$\frac{\omega}{(\log \omega)^\beta}\int_0^p \xi(t)S_{\beta-1}(\omega t)\, dt$$

$$= \frac{-\omega}{(\log \omega)^\beta}\Big[g(t)tS_{\beta-1}(\omega t)\Big]_0^p + \frac{\omega}{(\log \omega)^\beta}(\beta-1)\int_0^p g(t)S_{\beta-2}(\omega t)dt$$

$$= O[g(p)/\log \omega] + o(1) + \frac{\omega(\beta-1)}{(\log \omega)^\beta}\int_0^\pi g(t)S_{\beta-2}(\omega t)\, dt.$$

We also get

$$\frac{\omega}{(\log \omega)^\beta}\int_0^p \xi(t)S_\beta(\omega t)\, dt = O[g(p)] + o(1) + \frac{\omega\beta}{(\log \omega)^\beta}\int_0^\pi g(t)S_{\beta-1}(\omega t)\, dt.$$

Summing up above estimations, we see that

$$\text{(4.2)}\quad R_\beta(\omega) - s = \frac{2}{\pi}\frac{\omega}{(\log \omega)^\beta}\int_0^\pi g(t)[\beta(\beta-1)S_{\beta-2}(\omega t) - \beta S_{\beta-1}(\omega t)]dt + o(1),$$

for sufficiently large p and $\beta \geqq 2$.

Suppose $\beta > 2$ and let $h = [\beta-2]$, then, by h time application of integration by parts, we get

$$\int_0^\pi g(t)S_{\beta-2}(\omega t)\, dt = (\beta-2)(\beta-3)\cdots(\beta-2-h)\int_0^\pi S_{\beta-2-h-1}(\omega t)g_{h+1}(t)\, dt.$$

Using here the formula[2)]

$$g_{h+1}(t) = \frac{1}{\Gamma(h+1-\beta+2)}\int_t^\pi \left(\log\frac{u}{t}\right)^{h-\beta+2}\frac{g_{\beta-2}(u)}{u}du,$$

and (3.7), we have

$$\text{(4.3)}\quad \int_0^\pi g(t)S_{\beta-2}(\omega t)\, dt$$

$$= \frac{(\beta-2)(\beta-3)\cdots(\beta-2-h)}{\Gamma(h+1-\beta+2)}\int_0^\pi S_{\beta-h-3}(\omega t)dt\left\{\int_t^\pi \left(\log\frac{u}{t}\right)^{h-\beta+2}\frac{g_{\beta-2}(u)}{u}du\right\}$$

$$= \frac{(\beta-2)(\beta-3)\cdots(\beta-2-h)}{\Gamma(h+3-\beta)}\int_0^\pi \frac{g_{\beta-2}(u)}{u}du\int_0^u \left(\log\frac{u}{t}\right)^{h-\beta+2}S_{\beta-h-3}(\omega t)\, dt$$

2) *Cf.* Wang [6], Lemma 3.

$$=\Gamma(\beta-1)\int_0^\pi S_0(\omega u)g_{\beta-2}(u)\,du.$$

By similar estimation we get

$$\int_0^\pi g(t)\,S_{\beta-1}(\omega t)\,dt=\Gamma(\beta)\int_0^\pi S_0(\omega t)g_{\beta-1}(t)\,dt. \tag{4.4}$$

Substituting (4.3) and (4.4) into (4.2) and using (3.5), we get the following relation

$$R_\beta(\omega)-s=\frac{2}{\pi}\frac{\Gamma(\beta+1)}{(\log\omega)^\beta}\int_0^\pi\Big[g_{\beta-2}(t)-g_{\beta-1}(t)\Big]\frac{1-\cos\omega t}{t}\,dt+o(1). \tag{4.5}$$

Let us now put $\beta=\alpha+2\,(\alpha\geqq 0)$, then

$$R_{\alpha+2}(\omega)-s=\frac{2}{\pi}\frac{\Gamma(\alpha+3)}{(\log\omega)^{\alpha+2}}\int_0^\pi\Big[g_\alpha(t)-g_{\alpha+1}(t)\Big]\frac{1-\cos\omega t}{t}\,dt+o(1). \tag{4.6}$$

By the assumption of the theorem,

$$g_\alpha(t)=o\left[\left(\log\frac{1}{t}\right)^\alpha\right],\qquad g_{\alpha+1}(t)=o\left[\left(\log\frac{1}{t}\right)^{\alpha+1}\right],$$

as $t\to 0$. Hence, if we divide the integral (4.6) into those with the ranges $(0, 2/\omega)$ and $(2/\omega, \pi)$ and use the estimation $(1-\cos\omega t)/t=O(\omega)$ or $=O(1/t)$, we can easily get

$$R_{\alpha+2}(\omega)-s=o(1). \tag{4.7}$$

Thus Theorem 1 is completely proved.

(To be continued)

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