38. A Theorem of Dimension Theory

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Recently a dimension theory for general metric spaces has been established by M. Katětov and K. Morita.[1] The purpose of this note is to study some necessary and sufficient conditions for n-dimensionality of general metric spaces. In the present note we take the definition of dimension by H. Lebesgue or that by M. Katětov and K. Morita as the same: $\dim R=-1$ for $R=\phi$, $\dim R\leqq n$ if and only if for any pair of a closed set F and an open set G with $F\subseteq G$ there exists an open set U such that $F\subseteq U\subseteq G$, $\dim B(U)\leqq n-1$.[2]

Definition. For two collections $\mathfrak{U},\mathfrak{U}'$ of open sets we denote by $\mathfrak{U}<\mathfrak{U}'$ the fact that $U\subseteq U'$ for every $U\in\mathfrak{U}$ and for some $U'\in\mathfrak{U}'$.

Definition. We mean by a *disjoint collection* a collection $\mathfrak{U}$ of open sets such that $U,U'\in\mathfrak{U}$ and $U\neq U'$ imply $U\frown U'=\phi$.

Theorem 1. *In order that $\dim R\leqq n$ for a metric space R it is necessary and sufficient that there exist $n+1$ sequences $\mathfrak{U}_1^i>\mathfrak{U}_2^i>\cdots$ $(i=1,2,\cdots,n+1)$ of disjoint collections such that $\{\mathfrak{U}_m^i\mid i=1,\cdots,n+1;\ m=1,2,\cdots\}$ is an open basis of R.*

Proof. If $\dim R=0$,[3] then from M there exists a sequence $\mathfrak{V}_m$ $(m=1,2,\cdots)$ of locally finite coverings consisting of open, closed sets such that $S(p,\mathfrak{V}_m)$ $(m=1,2,\cdots)$[4] is a nbd (=neighbourhood) basis of each point p of R. For $\mathfrak{V}_m=\{V_\alpha\mid\alpha<\tau\}$ we define $\mathfrak{V}'_m=\{V_\alpha-\smile_{\beta<\alpha}V_\beta\mid\alpha<\tau\}$ and $\mathfrak{U}_1=\mathfrak{V}'_1$, $\mathfrak{U}_2=\mathfrak{U}_1\wedge\mathfrak{V}'_2$, $\mathfrak{U}_3=\mathfrak{U}_2\wedge\mathfrak{V}'_3,\cdots$. Then $\mathfrak{U}_1>\mathfrak{U}_2>\cdots$ is a sequence of disjoint collections, and $\{\mathfrak{U}_m\mid m=1,2,\cdots\}$ is an open basis of R.

Conversely, if there exists a sequence $\mathfrak{U}_1>\mathfrak{U}_2>\cdots$ of disjoint

1) M. Katětov: On the dimension of non-separable spaces. I, Czechoslovak Mathematical Journal, **2** (77), (1952). K. Morita: Normal families and dimension theory for metric spaces, Math. Annalen, **128** (1954); A condition for the metrizability of topological spaces and for n-dimensionality, Science Reports of the Tokyo Kyoiku Daigaku, Sect. A, **5**, No. 114 (1955).

2) $B(U)$ denotes the boundary of U. See K. Morita: Normal families and dimension theory for metric spaces; from now forth we call this paper M.

3) From now forth we assume $R\neq\phi$.

4) In this note we concern ourselves only with open coverings. We call $\mathfrak{V}$ a locally finite covering if every point of R has some neighbourhood intersecting only finitely many elements of $\mathfrak{V}$. $S(A,\mathfrak{V})=\smile\{V\mid V\in\mathfrak{V},\ V\frown A\neq\phi\}$ for $A\subseteq R$. Notations of this paper are chiefly due to J. W. Tukey: Convergence and uniformity in topology (1940).

collections such that $\{\mathfrak{U}_m \mid m=1,2,\cdots\}$ is an open basis of R, then for an arbitrary point p of R and for every $U \in \mathfrak{U}_m$, $p \in U$ implies $U \frown U' = \phi$ $(U \neq U' \in \mathfrak{U}_m)$, and $p \notin U$ for every $U \in \mathfrak{U}_m$ implies $\frown\{S(p, \mathfrak{U}_j) \mid j=1,\cdots, m-1;\ S(p,\mathfrak{U}_j) \neq \phi\} = p$ from the fact that $\{\mathfrak{U}_m \mid m=1,2,\cdots\}$ is an open basis of R. Hence each $\mathfrak{U}_m$ is locally finite and consists of open, closed sets, and hence $\dim R=0$ from M.

Now we proceed to n-dimensional cases. Let $\dim R \leqq n$, then we can decompose R into $n+1$ 0-dimensional spaces R_i $(i=1,\cdots,n+1)$ by the general decomposition theorem due to Katětov and Morita, *i.e.* $R=\smile_{i=1}^{n+1} R_i$, $\dim R_i=0$. Then there exists a sequence $\mathfrak{V}_1^i > \mathfrak{V}_2^i > \cdots$ of disjoint collections of R_i such that $\{\mathfrak{V}_m^i \mid m=1,2,\cdots\}$ is an open basis of R_i. As is obvious from the above discussion for 0-dimensional cases, we may assume that every $\mathfrak{V}_m^i$ covers R_i. We put $\mathfrak{V}_m^i = \{V_{\alpha m} \mid \alpha \in A\}$ and take the maximal positive number ε for each $x \in V_{\alpha m}$ such that $S_\varepsilon(x) \frown R_i \subseteq V_{\alpha m}$.[5] Furthermore, we define $\varepsilon(m,x) = \mathrm{Min}\left(\frac{1}{m}, \frac{\varepsilon}{2}\right)$, $U_{\alpha m} = \smile\{S_{\varepsilon(m,x)}(x) \mid x \in V_{\alpha m}\}$, $\mathfrak{U}_m^i = \{U_{\alpha m} \mid \alpha \in A\}$. Then it is obvious that $\mathfrak{U}_1^i > \mathfrak{U}_2^i > \cdots$ and each $\mathfrak{U}_m^i$ is a disjoint collection from the disjointness of $\mathfrak{V}_m^i$ and from the definition of $U_{\alpha m}$. Next we take an arbitrary point x of R and a positive number δ. We can take positive integers m, l such that $\frac{2}{m} < \delta$, $l \geqq m$, $x \in V_{\alpha l} \subseteq S_{1/m}(x)$ for some $V_{\alpha l} \in \mathfrak{V}_l^i$. Since for these integers $x \in U_{\alpha l} \subseteq S_{2/m}(x) \subseteq S_\delta(x)$ is obvious, we obtain an open basis $\{\mathfrak{U}_m^i \mid i=1,\cdots,n+1;\ m=1,2,\cdots\}$ of R.

Conversely, if R admits $n+1$ sequences $\mathfrak{U}_1^i > \mathfrak{U}_2^i > \cdots$ $(i=1,\cdots, n+1)$ such that $\{\mathfrak{U}_m^i \mid i=1,\cdots,n+1;\ m=1,2,\cdots\}$ is an open basis of R, then we define $R_i = \{x \mid S(x, \mathfrak{U}_m^i)\ (m=1,2,\cdots) \text{ is a nbd basis of } x\}$. Since $\mathfrak{U}_m^i$ is a disjoint collection of R_i and $\{\mathfrak{U}_m^i \mid m=1,2,\cdots\}$ is an open basis of R_i, $\dim R_i=0$. Hence we get $\dim R \leqq n$ from $R=\smile_{i=1}^{n+1} R_i$.

Theorem 2. *In order that a T. topological space R is a metrizable space with $\dim R \leqq n$ it is necessary and sufficient that there exists a sequence $\mathfrak{V}_1 > \mathfrak{V}_2^* > \mathfrak{V}_2 > \mathfrak{V}_3^* > \cdots$*[6] *of open coverings such that $S(p, \mathfrak{V}_m)$ $(m=1,2,\cdots)$ is a nbd basis for each point p of R and such that each set of $\mathfrak{V}_{m+1}$ intersects at most $n+1$ sets of $\mathfrak{V}_m$.*

Proof. Necessity. If R is a metric space with $\dim R \leqq n$, then $R=\smile_{i=1}^{n+1} R_i$ for some 0-dimensional spaces R_i $(i=1,\cdots,n+1)$. Let $\mathfrak{U}=\{U_\alpha \mid \alpha \in A\}$ be an arbitrary locally finite open covering of R, then there exists a disjoint covering $\mathfrak{V}_i = \{V_\alpha \mid \alpha \in A\}$ of R_i such

5) $S_\varepsilon(x) = \{y \mid \text{distance } (x,y) < \varepsilon\}$.

6) $\mathfrak{V}^* = \{S(V, \mathfrak{V}) \mid V \in \mathfrak{V}\}$.

that $V_\alpha \subseteq U_\alpha$. Defining $V'_\alpha = \smile\{S_{\varepsilon(x)/2}(x) \mid x \in V_\alpha\}$ for $\varepsilon(x)$ such that $R_i \frown S_{\varepsilon(x)}(x) \subseteq V_\alpha$, $S_{\varepsilon(x)}(x) \subseteq U_\alpha$, we get a disjoint collection $\mathfrak{B}'_i = \{V'_\alpha \mid \alpha \in A\}$ of R such that $\mathfrak{B}'_i < \mathfrak{U}$. Hence $\mathfrak{B}' = \bigcup_{i=1}^{n+1} \mathfrak{B}'_i$ is a locally finite covering of R with order of $\mathfrak{B}' \leqq n+1$ and is a refinement of $\mathfrak{U}$. Hence there exists an open covering $\mathfrak{B}'' = \{V''_\beta \mid \beta \in B\}$ of R such that $\overline{V''_\beta} \subseteq V'_\beta$ for $\mathfrak{B}' = \{V'_\beta \mid \beta \in B\}$. It is obvious that every point p of R has some nbd intersecting at most $n+1$ of sets belonging to $\mathfrak{B}''$; we call such a covering to be of *local order* $\leqq n+1$.

Now let $\mathfrak{U}_1 > \mathfrak{U}_2 > \cdots$ be a sequence of uniform coverings giving the uniform topology of R, then from the paracompactness[7] of R and from the above conclusion we get a refinement $\mathfrak{B}_1$ of $\mathfrak{U}_1$ of local order $\leqq n+1$. Furthermore, we get a refinement $\mathfrak{B}_2$ of $\mathfrak{U}_2$ such that $\mathfrak{B}_2^* < \mathfrak{B}_1$, each set of $\mathfrak{B}_2$ intersects at most $n+1$ sets of $\mathfrak{B}_1$ and such that local order of $\mathfrak{B}_2 \leqq n+1$. By repeating such processes we obtain a sequence $\mathfrak{B}_1 > \mathfrak{B}_2^* > \mathfrak{B}_2 > \mathfrak{B}_3^* > \cdots$ of open coverings such that $\mathfrak{B}_m < \mathfrak{U}_m$ and such that each set of $\mathfrak{B}_{m+1}$ intersects at most $n+1$ of sets belonging to $\mathfrak{B}_m$. Since $S(p, \mathfrak{U}_m)$ $(m=1,2,\cdots)$ is a nbd basis of p, $S(p, \mathfrak{B}_m)$ $(m=1,2,\cdots)$ is also a nbd basis of p, and hence the necessity is proved.

Sufficiency. The metrizability of such a space is obvious from Urysohn-Alexandroff's theorem. We divide the proof of n-dimensionality into three parts.

1. If $\mathfrak{B}_1 > \mathfrak{B}_2^* > \cdots$ is a sequence satisfying the condition of this theorem, then for each point p of R, $S^{n+2}(p, \mathfrak{B}_{m+1+n+2})$ intersects obviously at most $n+1$ sets of $\mathfrak{B}_m$.[8] Putting $\mathfrak{U}_m = \mathfrak{B}_{1+(m-1)(n+3)}$ $(m=1, 2,\cdots)$, we get a sequence $\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \mathfrak{U}_3^* > \cdots$ of open coverings such that $S(p, \mathfrak{U}_m)$ $(m=1,2,\cdots)$ is a nbd basis of $p \in R$ and such that each $S^{n+2}(p, \mathfrak{U}_{m+1})$ intersects at most $n+1$ sets of $\mathfrak{U}_m$.

Let $\mathfrak{U}_m = \{U_\alpha \mid \alpha < \tau\}$, then we can prove firstly that there exist open sets U^i_α such that $\bigcup_{i=1}^{n+1} U^i_\alpha \subseteq U_\alpha$, $U^i_\alpha \frown U^i_\beta = \phi$ for $\alpha \neq \beta$ and such that $U_\alpha \supseteq M \in \mathfrak{U}_{m+1}$ implies $M \subseteq U^i_\alpha$ for some U^i_α. To prove this we define U^i_α $(\alpha < \tau)$ by induction so that

1) $\bigcup_{i=1}^{n+1} U^i_\alpha \subseteq U_\alpha$, 2) $U^i_\alpha \frown U^i_\beta = \phi$ for $\beta < \alpha$, 3) $U_\alpha \supseteq M \in \mathfrak{U}_{m+1}$ implies $M \subseteq U^i_\alpha$ for some U^i_α, 4) $U^i_\alpha \frown W^{n-i+2}_\alpha = \phi$ $(i=1,\cdots,n+1)$, where we put $S^k_\alpha = \{p \mid S^k(p, \mathfrak{U}_{m+1})$ intersects some k sets of U_γ $(\gamma > \alpha)\}$ $(k=1, 2,\cdots,n+1)$ and $W^k_\alpha = S^k_\alpha \smile S^{k+1}_\alpha \smile \cdots \smile S^{n+1}_\alpha$.

For $\alpha = 0$ we define $U^1_0 = U_0$, $U^i_0 = \phi$ $(i=2,\cdots,n+1)$. Since $S(p, \mathfrak{U}_{m+1})$ intersects at most $n+1$ of U_α $(\alpha < \tau)$, $U^1_0 \frown W^{n+1}_0 = \phi$ is obvious,

7) Every fully normal space is paracompact by A. H. Stone: Paracompactness and product spaces, Bull. Amer. Math. Soc., **54** (1948).

8) $S^1(p, \mathfrak{B}) = S(p, \mathfrak{B})$, $S^{n+1}(p, \mathfrak{B}) = S(S^n(p, \mathfrak{B}), \mathfrak{B})$.

and the other three conditions, also, are obviously satisfied.

Let us assume that U_β^i are defined for $\beta<\alpha$, then putting $V_\alpha^i = \bigcup_{\beta<\alpha} U_\beta^i$ and $U_\alpha^i = U_\alpha - \overline{V}_\alpha^i \cup \overline{W}_\alpha^{n-i+2}$ $(i=1,2,\cdots,n+1)$, we get U_α^i satisfying 1)–4). Since the validities of 1), 2), 4) for U_α^i are clear, we prove 3) only. If $M\in \mathfrak{U}_{m+1}$ is an arbitrary set contained in U_α, then since every $S^{n+2}(p, \mathfrak{U}_{m+1})$ intersects at most $n+1$ sets of U_γ $(\gamma\geqq\alpha)$, $M\frown W_\alpha^{n+1}\subseteq U_\alpha\frown W_\alpha^{n+1}=U_\alpha\frown S_\alpha^{n+1}=\phi$ from the definition of S_α^{n+1}. Hence either $M\frown W_\alpha^i=\phi$ $(i=1,\cdots,n+1)$ or $M\frown W_\alpha^{n-i+2}\neq\phi$, $M\frown W_\alpha^{n-i+3}=\phi$ for some i such that $2\leqq n-i+3\leqq n+1$. If the former is the case, then $M\frown W_\alpha^1=\phi$. Since $U_\alpha\subseteq S_\beta^1\subseteq W_\beta^1$ is obvious for every $\beta<\alpha$, and since $U_\beta^{n+1}\frown W_\beta^1=\phi$ $(\beta<\alpha)$ from the assumption of induction, it holds $U_\alpha\frown U_\beta^{n+1}=\phi$ for every $\beta<\alpha$, and hence $U_\alpha\frown V_\alpha^{n+1}=\phi$. Thus we get $M\frown V_\alpha^{n+1}=\phi$ and consequently $M\subseteq U_\alpha^{n+1}$.

If the latter is the case, *i.e.* $y\in M\frown W_\alpha^{n-i+2}\neq\phi$, $M\frown W_\alpha^{n-i+3}=\phi$, $2\leqq n-i+3\leqq n+1$, then $y\in S_\alpha^{n-i+2+k}$ for some $k\geqq 0$, *i.e.* $S^{n-i+2+k}(y, \mathfrak{U}_{m+1})$ intersects some $n-i+2+k$ sets of U_γ $(\gamma>\alpha)$. Since $x, y\in M \in\mathfrak{U}_{m+1}$, $S^{n-i+2+k+1}(x, \mathfrak{U}_{m+1})$ intersects $n-i+2+k+1$ sets of U_γ $(\gamma\geqq\alpha)$. Hence $x\in S_\beta^{n-i+3+k}\subseteq W_\beta^{n-i+3}$ for every $\beta<\alpha$, and hence $M\subseteq W_\beta^{n-i+3}$. Since $U_\beta^{i-1}\frown W_\beta^{n-i+3}=\phi$ $(\beta<\alpha)$ from the assumption of induction, we get $M\frown U_\beta^{i-1}=\phi$ $(\beta<\alpha)$ and consequently $M\frown V_\alpha^{i-1}=\phi$. Combining this conclusion with the assumption $M\frown W_\alpha^{n-i+3}=\phi$, we obtain $M\frown(\overline{V}_\alpha^{i-1}\cup\overline{W}_\alpha^{n-i+3})=\phi$, *i.e.* $M\subseteq U_\alpha^{i-1}$. Thus the condition 3) is valid for α, and hence we can define U_α^i $(i=1,\cdots,n+1)$ satisfying 1)–3) for every $\alpha<\tau$.

2. Since $\mathfrak{U}_{m+2}^*<\mathfrak{U}_{m+1}<\{U_\alpha^i \mid i=1,\cdots,n+1;\alpha<\tau\}$, if we put $\mathfrak{U}_m^i=\{U_\alpha^i - \overline{S(R-\overline{U_\alpha^i}, \mathfrak{U}_{m+2})} \mid \alpha<\tau\}$, then $\bigcup_{i=1}^{n+1}\mathfrak{U}_m^i$ is an open covering refining $\mathfrak{U}_m$, and $U_1, U_2\in\mathfrak{U}_m^i$ and $U_1\neq U_2$ imply $S(U_1, \mathfrak{U}_{m+2})\frown S(U_2, \mathfrak{U}_{m+2})=\phi$. From now forth let us denote $\mathfrak{U}_{2m-1}$ by $\mathfrak{U}_m$ $(m=1,2,\cdots)$ for brevity, then $S(U_1, \mathfrak{U}_{m+1})\frown S(U_2, \mathfrak{U}_{m+1})=\phi$ if $U_1, U_2\in\mathfrak{U}_m^i$, $U_1\neq U_2$.

For every $U\in\mathfrak{U}_{2k-1}^i$ $(k=1,2,\cdots)$ we define inductively $\mathfrak{S}(U)=\mathfrak{S}^1(U)=\{U' \mid U'\in\mathfrak{U}_{2k-1+2j}^i$ for some natural number j, $S(U', \mathfrak{U}_{2k-1+2j})\frown U\neq\phi\}$, $\mathfrak{S}^{m+1}(U)=\bigcup\{\mathfrak{S}(U') \mid U'\in\mathfrak{S}^m(U)\}$ $(m=1,2,\cdots)$. (From now forth we denote by $U\leftarrow U'$ the fact that $S(U', \mathfrak{U}_{2k-1+2j})\frown U\neq\phi$ for $U'\in\mathfrak{U}_{2k-1+2j}^i$, $U\in\mathfrak{U}_{2k-1}^i$.) Furthermore, we define $S(U)=\bigcup\{U' | U'\in\bigcup_{m=1}^{\infty}\mathfrak{S}^m(U)\}$. The principal object of the second part is to prove i) $U_1, U_2\in\mathfrak{U}_{2k-1}^i$ and $U_1\neq U_2$ imply $S(U_1)\frown S(U_2)=\phi$, ii) $U_1\in\mathfrak{U}_{2k-1}^i$ and $U_2\in\mathfrak{U}_{2k-1+l}^i$ for some $l\geqq 2$ imply $S(U_2)\subseteq S(U_1)$ or $S(U_1)\frown S(U_2)=\phi$.

To prove i) we take an arbitrary $V\in\bigcup_{m=1}^{\infty}\mathfrak{S}^m(U)$. If $V\in\mathfrak{S}^j(U)$, then there exists a sequence $U_1=V_0\leftarrow V_1\leftarrow V_2\leftarrow\cdots\leftarrow V_j=V$ of $V_p\in\mathfrak{U}_{2k-1+n(p)}^i$ $(p=0,1,\cdots,j)$. We notice $n(p+1)\geqq n(p)+2$ and $n(1)\geqq 2$.

Since $\mathfrak{U}^*_{2k-1+n(p)} < \mathfrak{U}_{2k-1+n(p)-1}$, from the meaning of $\leftarrow$ combining with the above notice we see easily $V_j \subseteq S(V_j, \mathfrak{U}_{2k-1+n(j)}) \subseteq S(V_{j-1}, \mathfrak{U}_{2k-1+n(j-1)}) \subseteq S(V_{j-2}, \mathfrak{U}_{2k-1+n(j-2)}) \subseteq \cdots \subseteq S(V_1, \mathfrak{U}_{2k-1+n(1)}) \subseteq U'$ for some $U' \in \mathfrak{U}_{2k-1+1}$. Since $U' \frown U_1 \neq \phi$ from the fact $U_1 \leftarrow V_1$, we get $V_j \subseteq S(U_1, \mathfrak{U}_{2k-1+1})$. Therefore $S(U_1) \subseteq S(U_1, \mathfrak{U}_{2k-1+1})$. If $U_1, U_2 \in \mathfrak{U}^i_{2k-1}$ and $U_1 \neq U_2$, then since $S(U_1, \mathfrak{U}_{2k-1+1}) \frown S(U_2, \mathfrak{U}_{2k-1+1}) = \phi$, we can conclude $S(U_1) \frown S(U_2) = \phi$. As is easily seen from the above discussion, it holds $\{S(U) \mid U \in \mathfrak{U}^i_{2k-1}\} < \mathfrak{U}^*_{2k-1}$, which will be used later.

Next we proceed to the case of ii). If $S(U_1) \frown S(U_2) \neq \phi$ for $U_1 \in \mathfrak{U}^i_{2k-1}$ and $U_2 \in \mathfrak{U}^i_{2k-1+l}$, then there exist some $V_p \in \mathfrak{S}^p(U_1)$, $W_q \in \mathfrak{S}^q(U_2)$ with $V_p \frown W_q \neq \phi$ and consequently two sequences $U_1 = V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow \cdots \leftarrow V_p$, $U_2 = W_0 \leftarrow W_1 \leftarrow W_2 \leftarrow \cdots \leftarrow W_q$ of $V_j \in \mathfrak{U}^i_{2k-1+n(j)}$ $(j=0,1,\cdots,p)$ and of $W_j \in \mathfrak{U}^i_{2k-1+l+m(j)}$ $(j=0,1,\cdots,q)$ respectively. We take $j \geqq 0$ such that $2k-1+n(j) \leqq 2k-1+l < 2k-1+n(j+1)$; we notice $2k-1+l+2 \leqq 2k-1+n(j+1)$. Since $2k-1+n(j)=2k-1+l$ implies $S(U_2) = S(V_j) \subseteq S(U_1)$ from i), we assume $2k-1+n(j) < 2k-1+l$. If $2k-1+n(p) < 2k-1+l$, then since from the discussion of i) there exists $W' \in \mathfrak{U}_{2k-1+l+1}$ such that $W' \supseteq W_q$ and $W' \frown U_2 \neq \phi$, we get $V_p \leftarrow U_2$, and hence $S(U_2) \subseteq S(V_p) \subseteq S(U_1)$. If $j<p$, then from the discussion of i) there exist $W' \in \mathfrak{U}_{2k-1+l+1}$ and $V' \in \mathfrak{U}_{2k-1+n(j+1)+1}$ such that $W' \supseteq W_q$, $W' \frown U_2 \neq \phi$, $V' \supseteq V_p$, $V' \frown V_{j+1} \neq \phi$. Since $V_j \leftarrow V_{j+1}$, there exists $V'' \in \mathfrak{U}_{2k-1+n(j+1)}$ with $V'' \frown V_{j+1} \neq \phi$, $V'' \frown V_j \neq \phi$. Therefore $V' \smile V_{j+1} \smile V'' \in \mathfrak{U}^*_{2k-1+n(j+1)} < \mathfrak{U}_{2k-1+l+1}$ from the fact $2k-1+l+2 \leqq 2k-1+n(j+1)$. Since $W_q \frown V_p \subseteq W' \frown V' \neq \phi$, $W' \smile V' \smile V_{j+1} \smile V'' = W'' \in \mathfrak{U}^*_{2k-1+l+1} < \mathfrak{U}_{2k-1+l}$ and $W' \frown U_2 \subseteq W'' \frown U_2 \neq \phi$, $V'' \frown V_j \subseteq W'' \frown V_j \neq \phi$. Thus we conclude $V_j \leftarrow U_2$ and consequently $S(U_2) \subseteq S(V_j) \subseteq S(U_1)$.

3. Putting $\mathfrak{S}^i_m = \{S(U) \mid U \in \mathfrak{U}^i_{2m-1}\}$, we define inductively ${}_m\mathfrak{S}^i_{m+1} = \mathfrak{S}^i_m \smile \{S \mid S \in \mathfrak{S}^i_{m+1}, S \not\subseteq S'$ for every $S' \in \mathfrak{S}^i_m\}$, ${}_m\mathfrak{S}^i_{m+j+1} = {}_m\mathfrak{S}^i_{m+j} \smile \{S \mid S \in \mathfrak{S}^i_{m+j+1}, S \not\subseteq S'$ for every $S' \in {}_m\mathfrak{S}^i_{m+j}\}$ $(j=1,2,\cdots)$ for a fixed m. Then $\mathfrak{T}^i_m = \bigcup_{j=1}^{\infty} {}_m\mathfrak{S}^i_{m+j}$ is a disjoint collection from 2. Since ${}_{m+1}\mathfrak{S}^i_{m+1+j} < \bigcup_{k=0}^{j} \mathfrak{S}^i_{m+1+k} < {}_m\mathfrak{S}^i_{m+1+j}$, $\mathfrak{T}^i_{m+1} = \bigcup_{j=1}^{\infty} {}_{m+1}\mathfrak{S}^i_{m+1+j} < \bigcup_{j=2}^{\infty} {}_m\mathfrak{S}^i_{m+j} < \bigcup_{j=1}^{\infty} {}_m\mathfrak{S}^i_{m+j} = \mathfrak{T}^i_m$. Since $\bigcup_{i=1}^{n+1} \mathfrak{S}^i_m (> \mathfrak{U}_{2m})$ covers R and is a refinement of $\mathfrak{U}^*_{2m-1}$ from the remark at the end of the proof of 2-i). $S(p, \bigcup_{i=1}^{n+1} \mathfrak{S}^i_m)$ is a nbd basis for every point p of R; hence from $\mathfrak{S}^i_m \subseteq \mathfrak{T}^i_m$ it is obvious that $\{\mathfrak{T}^i_m \mid i=1,\cdots, n+1; m=1,2,\cdots\}$ is an open basis of R. Thus we get $n+1$ sequences $\mathfrak{T}^i_1 > \mathfrak{T}^i_2 > \cdots$ $(i=1,\cdots,n+1)$ of disjoint collections such that $\{\mathfrak{T}^i_m\}$ is an open basis of R. Therefore we conclude $\dim R \leqq n$ from Theorem 1.