88. Note on Algebras of Strongly Unbounded Representation Type¹⁾

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§1. Let A be an associative algebra with a unit element over an algebraically closed field k and $g_A(d)$ be the number of inequivalent indecomposable representations of A of degree d where d is a positive integer. Then if A has indecomposable representations of arbitrary high degrees and $g_A(d) = \infty$ for an infinite number of integers d, A is said to be of strongly unbounded representation type. In a paper [1] James P. Jans proved that the following four conditions are sufficient for an algebra to be of strongly unbounded representation type:

(1) L_A , the two-sided ideal lattice, is infinite.

(2) For any *i* and any two-sided ideal A_0 in N(N is the radical of A) e_iA_0 (A_0e_i) has more than three covers in $e_iN(Ne_i)$ where A' is said to be the cover of e_iA if $A' \supset e_iA_0$ and $A' \supset B \supseteq e_iA_0$ implies $B = e_iA_0$.

(3) The graph $G(A_0)$ associated with any two-sided ideal $A_0 \subset N$ is a cycle where the graph $G(A_0)$ is such a set $\{P_1, P_1 \& P_2, P_2, P_2 \& P_3, P_3, \cdots, P_{n-1}, P_{n-1} \& P_n, P_n\}^{2}$ that $P_i \& P_j$ holds if $e_i A'e_j$ covers $e_i A_0 e_j$ for some cover A' of A_0 and $G(A_0)$ is said to be the cycle if $\{G(A_0), G(A_0)\}$ is also a graph.

(4) The graph $G(A_0)$ associated with any two-sided ideal $A_0 \subset N$ branches at each end where $G_1(A_0)$ is said to extend $G_2(A_0)$ at the right end if $\{G_2(A_0), G_1(A_0)\}$ is the graph and $G(A_0)$ is said to branch at one end if it is extended by at least two distinct graphs at one end.

Now in this paper we shall prove that the following two conditions are also sufficient for an algebra to be of strongly unbounded representation type:

¹⁾ James P. Jans [1].

²⁾ P_1, P_2, \dots, P_n mean vertices, and " $P_i \& P_j$ " means that " P_i and P_j are connected by an (oriented) edge".

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§2. In this chapter we shall prove the following

Theorem 1. If the graph $G(A_0)$ associated with $A_0 \subset N$ is $\begin{cases} P_{r_2}, P_{k_1} \& P_{r_2}, P_{k_1}, P_{k_1} \& P_{r_1}, P_{r_1}, P_{k_3} \& P_{r_1}, P_{k_3}, P_{k_3} \& P_{r_4}, P_{r_4} \end{cases}$, then A $P_{r_3}, P_{k_2} \& P_{r_3}, P_{k_2}, P_{k_2} \& P_{r_1}, P_{r_2} \& P_{r_1}, P_{r_3} \otimes P_{r_4}, P_{r_4} \end{cases}$, then A is of strongly unbounded representation type.

Proof. In order to simplify the proof we assume that A is the basic algebra and $N^2=0$. In the case where $N^2 \neq 0$, we can prove by the same way as [1]. First we have the representation

$$R_{ij}\!\left(a
ight)\!=\!egin{bmatrix} X_{kj}\!\left(a
ight) \ Y_{ij}\!\left(a
ight) & X_{r_i}\!\left(a
ight) \end{bmatrix}$$

for P_{k_i} & P_{r_j} where $R_{ij}(a_{ij})$ has only 1 in the lower corner, namely $Y_{ij}(a_{ij})=1$ and $X_{s_i}(a_{ij})=0$, for $a_{ij} \in e_i Ne_j$ and $R_{ij}(a_{si})=0$ if $(i, j) \neq (s, t)$.

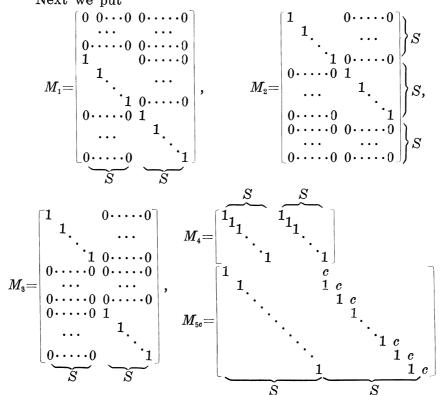
Next we construct the matrix function R_{cs} , $c \in k$ and S is an integer,

$$R_{cs}(a) = egin{cases} X_{T}(a) \ Y(a) & X_{B}(a) \end{bmatrix}$$
 ,

as follows.

Let $X_r(a)$ be the direct sum of $I_{2s}*X_{k_i}(a)$ (i=1, 2, 3) and let $X_B(a)$ be the direct sum of $I_{3s}*X_{r_1}(a)$ and $I_s*X_{r_i}(a)$ (i=2, 3, 4). Then it is clear that $X_r(a)$ and $X_B(a)$ are representations of A.

Next we put



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Let Y(a) have $M_1^*Y_{11}(a)$ directly below $I_{2s}^*X_{k_1}(a)$ and directly to the left of $I_{3s}^*X_{r_1}(a)$, $M_2^*Y_{12}(a)$ directly below $I_{2s}^*X_{k_2}(a)$ and directly to the left of $I_{3s}^*X_{r_1}(a)$, $M_3^*Y_{13}(a)$ directly below $I_{2s}^*X_{k_3}(a)$ and directly to the left of $I_{3s}^*X_{r_1}(a)$, $M_4^*Y_{21}(a)$ directly below $I_{2s}^*X_{k_1}(a)$ and directly to the left of $I_s^*X_{r_1}(a)$, $M_4^*Y_{21}(a)$ directly below $I_{2s}^*X_{k_1}(a)$ and directly to the left³ of $I_s^*X_{r_2}(a)$, $M_4^*Y_{32}(a)$ directly below $I_{2s}^*X_{k_2}(a)$ and directly to the left of $I_s^*X_{r_3}(a)$ and $M_5^*Y_{43}(a)$ directly below $I_{2s}^*X_{k_3}(a)$ and directly to the left of $I_s^*X_{r_4}(a)$. Fill out the rest with zeroes.

Now it is shown by the computation of eigenvalues of any commutator of $R_{cs}(a)$ that $R_{cs}(a)$ is a directly indecomposable representation.

Next we shall prove that A is of strongly unbounded representation type. For this purpose we have only to prove that $R_{cs}(a)$ and $R_{ds}(a)$ can not be similar for $c \neq d$. Now suppose they were similar. Then there would exist a non-singular matrix P intertwining R_{cs} and R_{ds} and we divide P into submatrices,

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{17} \\ P_{21} & P_{22} & \cdots & P_{27} \\ & & & \\ P_{71} & P_{72} & \cdots & P_{77} \end{bmatrix},$$

corresponding to the divisions of R_{cs} . It is clear from $R_{cs}(a)P = PR_{ds}(a)$ that $P_{1i} = 0$ $(i \neq 1)$, $P_{2i} = 0$ $(i \neq 2)$, $P_{3i} = 0$ $(i \neq 3)$ and P_{45} , P_{46} , P_{47} , P_{54} , P_{56} , P_{57} , P_{64} , P_{65} , P_{67} , P_{74} , P_{75} and P_{76} are zero, and $M_1P_{11} = P_{44}M_1$, $M_2P_{22} = P_{44}M_2$, $M_3P_{33} = P_{44}M_3$, $M_4P_{11} = P_{55}M_4$, $M_4P_{22} = P_{66}M_4$ and $M_{50}P_{33} =$ $P_{77}M_{56}$. Hence P_{11} , P_{22} , P_{33} and P_{44} are the direct sums of P_{77} and $P_{55} = P_{66} = P_{77}$ and from $M_{5c}P_{33} = P_{77}M_{5d}$, P_{77} can not be non-singular if $c \neq d.^{4}$)

§3. In this chapter we shall prove the following

Theorem 2. If the graph $G(A_0)$ associated with $A_0 \subset N$ is $\begin{cases} P_{k_5}, P_{k_5} \& P_{j_4}, P_{j_4}, P_{k_4} \& P_{j_4}, P_{k_4} \& P_{j_3}, P_{j_3}, P_{k_3} \& P_{j_3}, P_{k_3}, P_{k_3} \& P_{k_2}, P_{k_2} \& P_{j_2}, P_{j_2}, P_{j_2}, P_{k_1} \& P_{j_2}, P_{k_1} \& P_{j_1}, P_{j_1} \end{cases}$, then A is of strongly unbounded representation type.

Proof. By the same way as Theorem 1 we construct the matrix function R_{cs} , as follows,

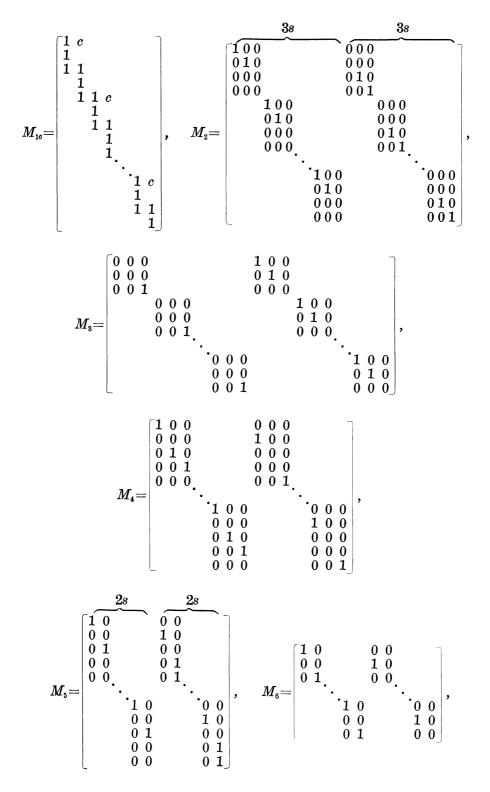
$$R_{cs}\!\left(a
ight)\!=\!\left\{\!egin{smallmatrix} X_{T}\!\left(a
ight)\ Y\!\left(a
ight) & X_{B}\!\left(a
ight)\!
ight\}egin{smallmatrix} egin{smallmatrix} X_{D}\!\left(a
ight)
ight\}egin{smallmatrix} egin{smallmatrix} egin{smal$$

Let $X_{r}(a)$ be the direct sum of $I_{2s}^{*}X_{j_{1}}(a)$, $I_{6s}^{*}X_{j_{2}}(a)$, $I_{4s}^{*}X_{j_{3}}(a)$ and $I_{2s}^{*}X_{j_{4}}(a)$ and let $X_{B}(a)$ be the direct sum of $I_{4s}^{*}X_{k_{1}}(a)$, $I_{3s}^{*}X_{k_{2}}(a)$, $I_{5s}^{*}X_{k_{3}}(a)$, $I_{3s}^{*}X_{k_{4}}(a)$ and $I_{s}^{*}X_{k_{5}}(a)$. Then $X_{r}(a)$ and $X_{B}(a)$ are all representations of A.

Next we put

^{3) *} means the Kronecker product.

⁴⁾ The computation is long and we shall omit it. Cf. T. Yoshii [2].



$$M_{7} = \begin{bmatrix} 3 & 3 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ \cdot & \cdot \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M_{8} = \begin{bmatrix} 1 & 0 \cdots 0 \\ 1 & \vdots & \vdots \\ \cdot & \vdots & \vdots \\ \cdot & 1 & 0 \cdots 0 \end{bmatrix}$$

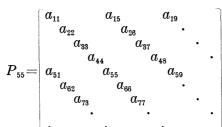
Then let Y(a) have $M_1^*Y_{11}$ directly below $I_{2s}^*X_{j_1}(a)$ and directly to the left of $I_{4s}^*X_{k_1}(a)$, $M_2^*Y_{12}(a)$ directly below $I_{6s}^*X_{j_2}(a)$ and directly to the left of $I_{4s}^*X_{k_1}(a)$, $M_3^*Y_{22}(a)$ directly below $I_{6s}^*X_{j_2}(a)$ and directly to the left of $I_{3s}^*X_{k_2}(a)$, $M_4^*Y_{22}(a)$ directly below $I_{6s}^*X_{j_2}(a)$ and directly to the left of $I_{5s}^*X_{k_3}(a)$, $M_5^*Y_{33}(a)$ directly below $I_{4s}^*X_{j_3}(a)$ and directly to the left of $I_{5s}^*X_{k_3}(a)$, $M_5^*Y_{33}(a)$ directly below $I_{4s}^*X_{j_3}(a)$ and directly to the left of $I_{5s}^*X_{k_3}(a)$, $M_6^*Y_{43}(a)$ directly below $I_{4s}^*X_{j_3}(a)$ and directly to the left of $I_{3s}^*X_{k_4}(a)$, $M_7^*Y_{44}(a)$ directly below $I_{2s}^*X_{j_4}(a)$ and directly to the left of $I_{3s}^*X_{k_4}(a)$ and $M_8^*Y_{54}(a)$ directly below $I_{2s}^*X_{j_4}(a)$ and directly to the left of $I_s^*X_{k_5}(a)$.

Now it is shown by the same way as above that R_{cs} is a directly indecomposable representation of A.

Next in order to show that A is of strongly unbounded representation type, we prove that R_{cs} and R_{ds} can not be similar for $c \neq d$. Now suppose they were similar. Then there would exist a non-singular matrix P intertwining R_{cs} and R_{ds} . Let P be divided into submatrices,

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \cdots P_{19} \\ P_{21} & P_{22} & P_{23} \cdots P_{29} \\ \vdots \\ P_{91} & P_{82} & P_{93} \cdots P_{99} \end{bmatrix},$$

corresponding to the divisions of R_{cs} . It is clear that P_{1i} $(i \neq 1)$, P_{2i} $(i \neq 2)$, P_{3i} $(i \neq 3)$, P_{4i} $(i \neq 4)$, P_{56} , P_{57} , P_{58} , P_{59} , P_{65} , P_{67} , P_{68} , P_{69} , P_{75} , P_{76} , P_{78} , P_{79} , P_{85} , P_{86} , P_{87} , P_{89} , P_{95} , P_{93} , P_{97} and P_{93} are zero. Moreover we have



and it is impossible from $PR_{cs}(a_{11}) = R_{ds}(a_{11})P$ that P is non-singular. Thus this proof is completed.

References

- [1] James P. Jans: On the Indecomposable Representation of Algebras, Dissertation, University of Michigan (1954).
- [2] T. Yoshii: On algebras of bounded representation type, Osaka Math. Jour., 8, No. 1 (1956) (forthcoming).