5. Some Properties of Completely Normal Spaces

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1. Recently, T. Inokuma [1] has given an answer to a problem proposed by A. D. Wallace [2] by proving that a topological space satisfies the condition of Wallace if and only if it is completely normal. Moreover, he has proved the following theorem:

If a topological space H is completely normal and fully normal, then for any locally finite family of subsets $X_1, X_2, \cdots$ of H, there exists a family of closed sets $H_1, H_2, \cdots$ satisfying the condition (V*):

$$(\mathrm{V}^*)\qquad \begin{cases} H=H_1\cup H_2\cup\cdots \\ H_i\cap H_j\cap(\overline{X}_i\cup\overline{X}_j)=\overline{X}_i\cap\overline{X}_j \\ \overline{X}_i\subset H_i \qquad i=1,2,\cdots . \end{cases}$$

In this paper we shall prove that in the above theorem the assumption of "full normality" is superfluous, and moreover, we shall establish some other related properties of completely normal spaces.

2. We shall first remark that the third relation of (V*) is derived from the second one; this is seen by putting $i=j$ in the second relation of (V*).

Secondly, without loss of generality we may assume that every X_i in the above theorem is a closed subset of H; so each X_i is assumed to be a closed subset of H in the following.

We can easily prove the following lemma by a simple calculation of sets.

Lemma 1. *Let H be a set, and let $\{X_\alpha \mid \alpha\in\Omega\}$ and $\{H_\alpha\mid\alpha\in\Omega\}$ be two families of subsets of the set H; then the following three conditions* (A), (B) *and* (C) *are equivalent*:

$$(\mathrm{A})\qquad \begin{cases} (\mathrm{A1})\quad H=\bigcup_{\alpha\in\Omega} H_\alpha \\ (\mathrm{A2})\quad H_\alpha\cap H_\beta\cap(X_\alpha\cup X_\beta)=X_\alpha\cap X_\beta, \qquad (\alpha,\beta\in\Omega) \end{cases}$$

$$(\mathrm{B})\qquad \begin{cases} (\mathrm{B1})=(\mathrm{A1}) \\ (\mathrm{B2})\quad H_\alpha\cap X_\beta=X_\alpha\cap X_\beta, \qquad (\alpha,\beta\in\Omega) \end{cases}$$

$$(\mathrm{C})\qquad \begin{cases} (\mathrm{C1})=(\mathrm{A1}) \\ (\mathrm{C2})\quad H_\alpha\cap(\bigcup_{\gamma\in\Omega} X_\gamma)=X_\alpha, \qquad (\alpha\in\Omega). \end{cases}$$

Since the condition (V*) is, of course, identical with the condition (A) of Lemma 1, we obtain the following theorem by T. Inokuma [1, Theorems 1, 2].

Theorem 1. *In order that a topological space H be a completely normal space, it is necessary and sufficient that for any finite family*

$\{X_1, X_2, \cdots, X_n\}$ *of closed subsets of* H *there exist a family* $\{H_1, H_2, \cdots, H_n\}$ *of closed subsets of* H *satisfying one of the equivalent conditions* (A), (B) *and* (C) *of Lemma 1.*

3. Now, we shall prove our main theorem.

Theorem 2. *If* H *is a completely normal space, then for any locally finite family* $\{X_\alpha | \alpha \in \Omega\}$ *of closed subsets of* H *there exists a family* $\{H_\alpha | \alpha \in \Omega\}$ *of closed subsets with the property* (A).

Proof. In case the cardinal number of Ω is finite, this theorem has already been established (Theorem 1); so we can assume that the cardinal number of Ω is infinite.

By Lemma 1 it is sufficient for the proof of this theorem to construct a family $\{H_\alpha | \alpha \in \Omega\}$ of closed subsets of H satisfying the condition (C) of Lemma 1. We assume that the set Ω of indices consists of all transfinite ordinals α less than η. Let us put $X'_\mu = X_{\mu+1} \cup X_{\mu+2} \cup \cdots$ for any $\mu < \eta$ and $X'_* = \bigcup_{\alpha \in \Omega} X_\alpha$. Since the family $\{X_\alpha\}$ is a locally finite family, for any $\mu < \eta$ each X'_ι is a closed subset of H.

Let ν be an ordinal such that $\nu < \eta$. We shall assume that for every $\mu < \nu$ there exist H_μ, H'_μ of closed subsets of H satisfying the condition:

$$(\mathrm{P}_\mu) \qquad \begin{cases} (\bigcup_{\gamma \leq \mu} H_\gamma) \cup H'_\mu = H \\ H_\mu \cap X'_* = X_\mu, \quad H'_\mu \cap X'_* = X'_\mu. \end{cases}$$

Then we shall construct two closed subsets H_ν, H'_ν of H satisfying the condition (P_ν).

Put

$$(1) \qquad H''_\nu = \bigcap_{\mu < \nu} H'_\mu.$$

Since H''_ν is a closed subset of H containing two subsets X_ν and X'_ν, and H is a completely normal space, for X_ν and X'_ν there exist two closed subsets H_ν and H'_ν of H''_ν (and hence of H) such that

$$(2) \qquad H''_\nu = H_\nu \cup H'_\nu$$

and

$$(3) \qquad H_\nu \cap (X_\nu \cup X'_\nu) = X_\nu, \quad H'_\nu \cap (X_\nu \cup X'_\nu) = X'_\nu.$$

Then we shall show that the two closed sets H_ν and H'_ν satisfy the condition (P_ν).

By (1) we get $H - H''_\nu = H - \bigcap_{\mu < \nu} H'_\mu = \bigcup_{\mu < \nu} (H - H'_\mu) \subset \bigcup_{\mu < \nu} H_\mu$. Hence, by (2) we get $(\bigcup_{\mu \leq \nu} H_\nu) \cup H'_\nu = H$. Next, we obtain $H_\nu \cap X'_* = H_\nu \cap H''_\nu \cap X'_* = H_\nu \cap (\bigcap_{\mu < \nu} H'_\mu) \cap X'_* = H_\nu \cap (\bigcap_{\mu < \nu} (H'_\mu \cap X'_*)) = H_\nu \cap (\bigcap_{\mu < \nu} X'_\mu) = H_\nu \cap (X_\nu \cup X'_\nu) = X_\nu$. Similary, we get $H'_\nu \cap X'_* = H'_\nu \cap H''_\nu \cap X'_* = X'_\nu$. Consequently, all the conditions of (P_ν) are satisfied. Hence we know from the second condition of (P_ν) that the family of closed sets $\{H_\alpha | \alpha \in \Omega\}$ of H satisfies the condition (C2) of Lemma 1.

Now, let us put $D=\bigcap_{\alpha\in\Omega} H'_\alpha$. We distinguish two cases to construct a family satisfying the condition (C1) of Lemma 1.

Case 1) $D=0$. In this case we can easily see that the family $\{H_\alpha \mid \alpha\in\Omega\}$ satisfies the condition (C1) of Lemma 1.

Case 2) $D\neq 0$. In this case we shall first prove that for any $\beta\in\Omega$, the intersection of X_β and D is an empty set. Contrary to this assertion, suppose that for some $\beta\in\Omega$ the intersection of X_β and D is not empty and contains a point x of H. Then, since the family $\{X_\alpha\}$ is locally finite, it is, of course, a point-finite family; let ν be the greatest index such that X_ν contains the point x. Then for any $\mu>\nu$ the set X'_μ does not contain the point x. But we get $x\in X_\beta\cap D\subset X_\beta\cap H'_\mu \subset H'_\mu\cap X'_*=X'_\mu$. This contradicts the relation $x\notin X'_\mu$.

We put $H_1^\circ=H_1\cup D$; then we shall prove that the family $\mathfrak{H}=\{H_1^\circ, H_2,\cdots, H_\nu,\cdots\}$ satisfies the condition (C) of Lemma 1.

We can easily see that the family $\mathfrak{H}$ satisfies the relation (C2) of Lemma 1, since $H_1^\circ\cap X'_*=(H_1\cup D)\cap X'_*=(H_1\cap X'_*)\cup(D\cap X'_*)=H_1\cap X'_* =X_1$. (C1) of Lemma 1 will be proved for the family $\mathfrak{H}$ by the following relations: $H_1^\circ\cup H_2\cup\cdots\cup H_\nu\cup\cdots=(\bigcup_{\alpha\in\Omega} H_\alpha)\cup D=(\bigcup_{\alpha\in\Omega} H_\alpha)\cup(\bigcap_{\mu\in\Omega} H'_\mu) =\bigcap_{\mu\in\Omega}((\bigcup_{\alpha\in\Omega} H_\alpha)\cup H'_\mu)\supset\bigcap_{\mu\in\Omega}((\bigcup_{\alpha\leq\mu} H_\alpha)\cup H'_\mu)=\bigcap_{\mu\in\Omega} H=H.$

We have thus constructed the family satisfying the condition (C) of Lemma 1 in both cases $D=0$ and $D\neq 0$, q.e.d.

4. We shall show some more properties of completely normal spaces in the following

Theorem 3. *Let $\{X_1, X_2,\cdots\}$ be any locally finite countable family of closed subsets of a topological space H. In order that the space H be a completely normal space it is necessary and sufficient that there exist a countable family of closed sets $\{K, H_1, H_2,\cdots\}$ satisfying the following conditions*:

1) $$(\bigcup_{i=1}^{\infty} H_i)\cup K=H$$

2) $$H_i\cap(\bigcup_{j=1}^{\infty} X_j)=X_i \qquad \text{for } i=1,2,\cdots$$

3) $$H_i\cap H_j=X_i\cap X_j \qquad \text{for } i\neq j$$

4) $$K\cap(\bigcup_{i=1}^{\infty} X_i)=\bigcup_{i=1}^{\infty}(\bigcup_{j\neq i}(X_i\cap X_j)).$$

Proof. Since the sufficiency is evident, we prove only the necessity. Put $X'_i=X_{i+1}\cup X_{i+2}\cup\cdots$, $i=0,1,2,\cdots$. We assume that for $i-1$ there exist two closed subsets H_{i-1}, H'_{i-1} satisfying the relations:

$$(\mathrm{Q}_{i-1}1)\quad\begin{cases} H_{i-1}\cup H'_{i-1}=H'_{i-2} \\ H_{i-1}\cap(\bigcup_{j\leq i-2}(H_j\cap H'_j)\cup X'_{i-2})=X_{i-1} \\ H'_{i-1}\cap(\bigcup_{j\leq i-2}(H_j\cap H'_j)\cup X'_{i-2})=\bigcup_{j\leq i-2}(H_j\cap H'_j)\cup X'_{i-1}\end{cases}$$

$(\mathrm{Q}_{i-1}2)$ $$H_1\smile H_2\smile\cdots\smile H_{i-1}\smile H'_{i-1}=H$$
$(\mathrm{Q}_{i-1}3)$ $$H_j\frown X'_0=X_j \qquad j\leqq i-1$$
$(\mathrm{Q}_{i-1}4)$ $$H_j\frown H_k=X_j\frown X_k \qquad j\neq k \qquad j,k\leqq i-1$$
$(\mathrm{Q}_{i-1}5)$ $$H'_{i-1}\frown X'_0=X'_{i-1}\smile\{\bigcup_{j\leqq i-1}(\bigcup_{k\neq j}(X_j\frown X_k))\}.$$

Then we shall construct two closed subsets H_i, H'_i satisfying the conditions (Q_i1)–(Q_i5).

Since subsets X_i and $\bigcup_{j\leqq i-1}(H_j\frown H'_j)\smile X'_i$ are closed subsets of the completely normal space H'_{i-1}, there exist two closed subsets H_i and H'_i of H'_{i-1}, hence of H such that

(Q_i1) $$\begin{cases} H_i\smile H'_i=H'_{i-1} \\ H_i\frown(\bigcup_{j\leqq i-1}(H_j\frown H'_j)\smile X'_{i-1})=X_i \\ H'_i\frown(\bigcup_{j\leqq i-1}(H_j\frown H'_j)\smile X'_{i-1})=\bigcup_{j\leqq i-1}(H_j\frown H'_j)\smile X'_i. \end{cases}$$

Hence the condition (Q_i2) is evidently satisfied. Also we obtain the relation (Q_i3) from the following: $H_i\frown X'_0=H_i\frown H'_{i-1}\frown X'_0=H_i\frown[X'_{i-1}\smile\{\bigcup_{j\leqq i-1}(\bigcup_{k\neq j}(X_j\frown X_k))\}]\subset H_i\frown[X'_{i-1}\smile\{\bigcup_{j\leqq i-1}(H_j\frown H'_j)\}]=X_i$. On the other hand, the validity of the condition $H_i\frown X'_0\supset X_i$ is evident. Hence $H_i\frown X'_0=X_i$. From this and the assumption of induction we obtain:

(Q_i3) $$H_j\frown X'_0=X_j \qquad \text{for } j\leqq i.$$

Next, to prove the relation (Q_i4) we shall proceed as follows. Since

$$X_i=H_i\frown\{\bigcup_{j\leqq i-1}(H_j\frown H'_j)\smile X'_{i-1}\}=[H_i\frown\{\bigcup_{j\leqq i-1}(H_j\frown H'_j)\}]\smile\{H_i\frown X'_{i-1}\},$$

we have

$$X_i\supset H_i\frown\{\bigcup_{j\leqq i-1}(H_j\frown H'_j)\}=\bigcup_{j\leqq i-1}(H_i\frown H_j\frown H'_j)=\bigcup_{j\leqq i-1}(H_i\frown H_j).$$

Hence, for $j\leqq i-1$ the set X_i contains $H_i\frown H_j$. Therefore we obtain

$$H_j\frown H_i=H_j\frown H_i\frown X_i=H_j\frown X'_0\frown H_i\frown X_i=X_j\frown H_i\frown X_i=X_j\frown X_i \quad \text{for } j\leqq i-1.$$

From this and the assumption of induction we obtain:

(Q_i4) $$H_j\frown H_i=X_j\frown X_i \qquad j\neq k \qquad j,k\leqq i.$$

Finally, we shall treat the condition (Q_i5);

$$\begin{aligned} H'_i\frown X'_0&=H'_i\frown H'_{i-1}\frown X'_0 \\ &=H'_i\frown[X'_{i-1}\smile\{\bigcup_{j\leqq i-1}(\bigcup_{k\neq j}(X_j\frown X_k))\}] \\ &=H'_i\frown[\bigcup_{j\leqq i-1}(H_j\frown H'_j)\smile X'_{i-1}]\frown[X'_{i-1}\smile\{\bigcup_{j\leqq i-1}(\bigcup_{k\neq j}(X_j\frown X_k))\}] \\ &=[\bigcup_{j\leqq i-1}(H_j\frown H'_j)\smile X'_i]\frown[X'_{i-1}\smile\{\bigcup_{j\leqq i-1}(\bigcup_{k\neq j}(X_j\frown X_k))\}] \\ &=[\bigcup_{j\leqq i-1}(H_j\frown H'_j)\frown[X'_{i-1}\smile\{\bigcup_{j\leqq i-1}(\bigcup_{k\neq j}(X_j\frown X_k))\}]]\smile X'_i \\ &=[\{\bigcup_{j\leqq i-1}(H_j\frown H'_j)\}\frown X'_{i-1}]\smile[\{\bigcup_{j\leqq i-1}(H_j\frown H'_j)\} \\ &\qquad\frown\{\bigcup_{j\leqq i-1}(\bigcup_{k\neq j}(X_j\frown X_k))\}]\smile X'_i \\ &=[\{\bigcup_{j\leqq i-1}(H_j\frown H'_j)\}\frown X'_{i-1}]\smile[\bigcup_{j\leqq i-1}(\bigcup_{k\neq j}(X_j\frown X_k))]\smile X'_i. \end{aligned}$$

Now the first term of the last expression is transformed as following:

$$\{\bigcup_{j\leq i-1}(H_j\frown H_j')\}\frown X_{i-1}'=\bigcup_{j\leq i-1}(H_j\frown H_j'\frown X_{i-1}')=\bigcup_{j\leq i-1}(H_j\frown X_0'\frown H_j'\frown X_{i-1}')$$
$$=\bigcup_{j\leq i-1}(X_j\frown H_j'\frown X_{i-1}')=\bigcup_{j\leq i-1}(X_j\frown X_{i-1}').$$

Hence,

$$H_i'\frown X_0'=[\bigcup_{j\leq i-1}(X_j\frown X_{i-1}')]\smile[\bigcup_{j\leq i-1}(\bigcup_{k\neq j}(X_j\frown X_k))]\smile X_i'$$
$$=X_i'\smile[\bigcup_{j\leq i}(\bigcup_{k\neq j}(X_j\frown X_k))].$$

Thus (Q_i5) is satisfied.

Put $K=\bigcap_{i=1}^{\infty}H_i'$, then the family $\{K, H_1, H_2,\cdots\}$ will be proved to satisfy the conditions of the theorem. The conditions 2) and 3) are obvious, so we prove the conditions 1) and 4).

$$K\smile H_1\smile H_2\smile\cdots=(\bigcap_{i=1}^{\infty}H_i')\smile(\bigcup_{j=1}^{\infty}H_j)=\bigcap_{i=1}^{\infty}((\bigcup_{j=1}^{\infty}H_j)\smile H_i')$$
$$\subset\bigcap_{i=1}^{\infty}((\bigcup_{j\leq i}H_j)\smile H_i')=\bigcap_{i=1}^{\infty}H=H.$$

Hence the condition 1) is proved. By the definition of K,

$$K\frown X_0'=(\bigcap_{i=1}^{\infty}H_i')\frown X_0'=\bigcap_{i=1}^{\infty}(H_i'\frown X_0')$$
$$=\bigcap_{i=1}^{\infty}[X_i'\smile\{\bigcup_{j\leq i}(\bigcup_{k\neq j}(X_j\frown X_k))\}].$$

Since

$$X_i'\smile\{\bigcup_{j\leq i}(\bigcup_{k\neq j}(X_j\frown X_k))\}\supset X_{i+1}'\smile\{\bigcup_{j\leq i+1}(\bigcup_{k\neq j}(X_j\frown X_k))\},$$

we get the condition 4):

$$K\frown X_0'=\bigcup_{i=1}^{\infty}(\bigcup_{j\neq i}(X_i\frown X_j)).$$

This completes the proof of Theorem 3.

Theorem 4. *Let X_1, X_2 be any closed subsets of a T_1-space H, and K_1, K_2 be any open subsets with $K_1\supset X_1$, $K_2\supset X_2$. Then in order that the space H be a completely normal space it is necessary and sufficient that there exist two closed subsets H_1 and H_2 such that*

$$(\mathrm{F})\qquad\begin{cases}(\mathrm{F1}) & H_1\smile H_2=\overline{K}_1\smile\overline{K}_2\\ (\mathrm{F2}) & H_i\subset\overline{K}_i\qquad i=1,2\\ (\mathrm{F3}) & H_i\frown X_j=X_i\frown X_j\qquad i=1,2.\end{cases}$$

We shall first prove the following lemma.

Lemma 2. *Let H be a completely normal space (we shall not assume here that H is a T_1-space). Then there exist two closed subsets H_1 and H_2 satisfying the condition* (F) *of Theorem 4 under the same assumption as in Theorem 4.*

Proof. Put $F_1=\overline{K}_1-K_2$ and $F_2=\overline{K}_2-K_1$; then F_1 and F_2 are closed sets. If the intersection of $\overline{K}_1\frown\overline{K}_2$ is empty, the sets $\overline{K}_1=H_1$ and $\overline{K}_2=H_2$ satisfy the condition of the theorem. Next we shall assume that the intersection of $\overline{K}_1$ and $\overline{K}_2$ is not empty. Since $\overline{K}_1 \frown \overline{K}_2 \frown X_1$ and $\overline{K}_1 \frown \overline{K}_2 \frown X_2$ are closed in $\overline{K}_1 \frown \overline{K}_2$, there exist two closed subsets E_1 and E_2 of $\overline{K}_1 \frown \overline{K}_2$, hence of H, such that $E_1 \smile E_2 = \overline{K}_1 \frown \overline{K}_2$ and $E_i \frown \overline{K}_1 \frown \overline{K}_2 \frown X_j = \overline{K}_1 \frown \overline{K}_2 \frown X_i \frown X_j$ $(i, j=1, 2)$. Then if we put $H_i = F_i \smile E_i$ $(i=1, 2)$, we can easily prove that the sets H_1 and H_2 satisfy the condition (F).

Proof of Theorem 4. The necessity is obvious from the above lemma. We shall show the sufficiency: for any closed subsets X_1, X_2 there exist two closed subsets H_1 and H_2 such that $H_1 \smile H_2 = H$ and $H_i \frown X_j = X_i \frown X_j$. If a family $\{X_1, X_2\}$ covers the space H, $H_i = X_i$ $(i=1, 2)$ satisfy the above conditions. So we assume that the family $\{X_1, X_2\}$ does not cover H and X_1 differs from X_2. Without loss of generality we can assume that there is a point p such that $X_1 \ni p$ and $X_2 \not\ni p$. Next we take $q \notin X_1 \smile X_2$. Then $H - \{p\}$ and $H - \{q\}$ are open. Put $K_1 = H - \{q\}$, $K_2 = H - \{p\}$; then K_1 and K_2 satisty the condition (F) of Theorem 4. Hence there exist two closed subsets H_1 and H_2 such that $H_1 \smile H_2 = \overline{K}_1 \smile \overline{K}_2 = H$ and $H_i \frown X_j = X_i \frown X_j$ $(i, j=1, 2)$. This completes the proof of Theorem 4.

Theorem 5. *If for any locally finite family $\{X_\alpha \mid \alpha \in \Omega\}$ of closed subsets of a topological space H there exists a locally finite family $\{H_\alpha \mid \alpha \in \Omega\}$ of closed subsets of the space H with the property* (B) *of Lemma 1, then the space H is completely normal and collectionwise normal.*

Proof. It is sufficient to prove only the collectionwise normality. Let $\{X_\alpha \mid \alpha \in \Omega\}$ be a locally finite family of closed subsets of the space, whose elements are mutually disjoint. Let $\{H_\alpha \mid \alpha \in \Omega\}$ be a family satisfying the condition of this theorem. If we put $G_\alpha = H - \bigcup_{\beta \neq \alpha} H_\beta$, then G_α contains H_α and is open in H, and moreover, for $\alpha \neq \alpha'$ we get $G_\alpha \frown G_{\alpha'} = (H - \bigcup_{\beta \neq \alpha} H_\beta) \frown (H - \bigcup_{\beta \neq \alpha'} H_\beta) = H - ((\bigcup_{\beta \neq \alpha} H_\beta) \smile (\bigcup_{\beta \neq \alpha'} H_\beta))$
$= H - (\bigcup_{\alpha \in \Omega} H_\alpha) = 0.$

This completes the proof of Theorem 5.

References

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