26. *The Initial Value Problem for Linear Partial Differential Equations with Variable Coefficients.* II

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In the present note we give another simple proof of Theorem 2 in Paper I using the duality of Hilbert space defined over $R_t^1 \times R_x^l$. This idea owes to Prof. M. Nagumo who develops our theorem in an abstract form,[1] but it seems good to me to refine his postulate.

Let $A\left(t, x, \frac{\partial}{\partial x}\right)$ be an (m, m)-smooth system defined on $R_t^1 \times R_x^l$ and let $B\left(t, x, \frac{\partial}{\partial x}\right)$ be an other (m, m)-smooth system which is uniformly strongly elliptic with sufficiently large $s=\{s(i) \mid i=1, 2, \cdots, m\}$ for any $t \in R_t^1$.

Lemma 1. *For sufficiently large integers s' and s'' let $\mathfrak{D}_t^{(s')}(H_x^{(s'')})$ be a space of all s'-time differentiable vector valued functions on R_t^1 into $H_x^{(s'')}$*[2] *with compact carriers. Then for some integer $k(s'')$ the differential operator $B\left(t, x, \frac{\partial}{\partial x}\right)+B\left(t, x, \frac{\partial}{\partial x}\right)^*+k(s'')$ has the inverse from $\mathfrak{D}_{t,x}$ into $\mathfrak{D}_t^{(s')}(H_x^{(s'')})$.*

From Lemma 1 and Sobolev's lemma $(B+B^*+k(s''))^{-1}(\mathfrak{D}_{t,x})$ is contained in the space of functions defined over $R_t^1 \times R_x^l$ with derivatives of orders s for some $s < s' \wedge s''$.

Lemma 2. *Let A be semi-bounded by the norm defined by B in the strong sense, i.e.,*

$$((A_t u, u))_{B_t} \leqq \gamma((u, u))_{B_t} \qquad \text{for } u \in \mathfrak{D}_x$$

for some positive constant γ. Then for any $u \in \mathfrak{D}_{t,x}$ the following inequalities hold:

$$(1) \qquad \int_{-\infty}^{\infty} \left(\left(\left(\frac{\partial}{\partial t}-\bar{A}_t\right)u_t, u_t\right)\right)_{B_t} dt \geqq \beta \int_{-\infty}^{\infty} ((u_t, u_t))_{B_t} dt,$$

$$(2) \qquad \| e^{\delta t} u_t \|_{B_t} \leqq \| e^{\delta t_0} u_{t_0} \|_{B_{t_0}} + \left\{ \int_{t_0}^{t} \left\| e^{\delta \tau}\left(\frac{\partial}{\partial \tau}-\bar{A}_\tau\right)u_\tau \right\|_{B_\tau} d\tau \right\} \quad \text{for } t > t_0$$

where $\bar{A}=A-\alpha$, α, β, δ are some positive reals.

The inequality (1) implies the following:

$$(3) \qquad \int_{-\infty}^{\infty} \left\| \left(\frac{\partial}{\partial t}-\bar{A}\right)u_t \right\|_{B_t}^2 dt \geqq \beta^2 \int_{-\infty}^{\infty} \| u_t \|_{B_t}^2 dt$$

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1) M. Nagumo: On linear hyperbolic system of partial differential equations in the whole space, Proc. Japan Acad., **32** (1956).

2) $H_x^{(s)}$ is H_s in Paper I.

$$(4) \qquad \int_{-\infty}^{\infty} \left\| B_t^{-1}\left(\frac{\partial}{\partial t} - \bar{A}\right)^* B_t u_t \right\|_{B_t}^2 dt \geqq \beta^2 \int_{-\infty}^{\infty} \|u_t\|_{B_t}^2 dt$$

for any $u \in \mathfrak{D}_{t,x}$.

From Inequality (4) and Lemma 1, using the duality of Hilbert space we see the following

Lemma 3. *Let $L_t^{(2)}(H_x^{(s)})$ be the space of square integrable functions on R_t^1 with values in $H_x^{(s)}$. Then for any $v \in L_t^{(2)}(H^{(s)})$ there is a $u \in L_t^{(2)}(H_x^{(s)})$ such that*

$$(*) \qquad \left(\frac{\partial}{\partial t} - \bar{A}\right) u = v$$

in the sense of distributions in $R_t^1 \times R_x^l$.

Let $H_{t,x}^{(s)}$ be the Hibert space of all functions defined on $R_t^1 \times R_x^l$ with strong derivatives of orders $\leqq s$. Then we see the following

Lemma 4. *Let u be an element of $L_t^{(2)}(H_x^{(s)})$ and let v be an element of $H_{t,x}^{(s)}$ such that they satisfy Equation $(*)$ in the sense of distributions. Then u has strongly derivatives of orders $\leq s'$, i.e., $u \in H_{(t,x)}^{(s')}$ where s' depends on s and the order of A.*

For in the sense of distributions

$$\frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_l}}{\partial x^{i_l}} \frac{\partial^n}{\partial t^n} u \quad (n + i_1 + \cdots + i_l \leqq s')$$

is the form of the sum of derivatives with respect to x of $\left(\frac{\partial^j}{\partial t^j} A\right) u$ and $\frac{\partial^k}{\partial t^k} v$ which are in $L_{t,x}^{(2)}$, where $\frac{\partial^j}{\partial t^j} \bar{A}$ is the differential operator such that $\left(\frac{\partial^j}{\partial t^j} \bar{A}\right)(t, x, \xi) = \frac{\partial^j}{\partial t^j} (\bar{A}(t, x, \xi))$.

Therefore by Sobolev's lemma such solution u has ordinary derivatives of orders $\leqq s''$ $(< s')$, and thus from (2) and (3) we see the following

Lemma 5. *If $A\left(t, x, \frac{\partial}{\partial x}\right)$ is semi-bounded by the norm defined by $B^{(s_1)}$ and $B^{(s_2)}$ in the strong sense such that $s_1(i)$ is sufficiently larger than $s_2(i)$ for any i $(i = 1, 2, \cdots, m)$. Then for any $v \in H_{t,x}^{s'}$ (s' is a sufficiently large integer) there is uniquely a solution u of $(*)$ such that $u \in H_{t,x}^{s''}$, $s'' > s_2(i)$ $(i = 1, 2, \cdots, m)$. In particular if $v(t) = 0$ for $t \leqq 0$, then $u(t) = 0$ for $t \leqq 0$.*

From Lemma 5 we see Theorem 2.

Finally we remark that Theorem 2 is strengthened with respect to the condition of the coefficients of $A\left(t, x, \frac{\partial}{\partial x}\right)$ in our example given in Paper I by a limit process, and that Theorem 2 (Lemma 4, too) implies the hypoellipticity of parabolic equations in the more general sense than Petrovsky's.