84. On Closed Subspaces of the Complete Ranked Spaces

By Hatsuo OKANO

(Comm. by K. KUNUGI, M.J.A., June 12, 1957)

The purpose of this note is to study the completeness of the subspaces of the complete ranked spaces.[1]

Let R be a ranked space.[2] For a subset A, $\bar{A}$ denotes the closure of A. And for a fundamental sequence $v=\{v_\alpha(p_\alpha),\ 0\leq\alpha<\omega_\mu\}$, $\vartheta(v)$ denotes $\bigcap_\alpha I\{v_\alpha(p_\alpha)\}$ if $\omega_\mu<\omega_\nu$ and $\bigcap_\alpha v_\alpha(p_\alpha)$ if $\omega_\mu=\omega_\nu$. Now we shall introduce another topology as follows: for a subset A of R, let $\tilde{A}$ denote the set of all points p such that $p\in\vartheta(v)$ for some fundamental sequences $v=\{v_\alpha(p_\alpha)\}$, where $p_\alpha\in A$. Then the following conditions are satisfied:[3]

(1) $A\subseteq\tilde{A}$.

(2) If $A\subseteq B$, then $\tilde{A}\subseteq\tilde{B}$.

(3) $\tilde{0}=0$.[4]

(4) $\widetilde{A\cup B}\subseteq\tilde{A}\cup\tilde{B}$.

Take $\tilde{A}$ for the closure of A, and we get a new topology. We shall call it r-topology of R.

Let S be a subset of R, then, for the usual relative topology, we have $\omega(S)\geq\omega(R)$. So $\omega_\nu\leq\omega(S)$. Hence we can take, as $\mathfrak{B}_\alpha(0\leq\alpha<\omega_\nu)$, the set of all neighbourhoods of the form $v(p)=S\frown V(p)$, where $p\in S$ and $V(p)\in\mathfrak{B}_\alpha$ in R. Then the axiom (a) is satisfied and hence S is a ranked space.

The closed (or r-closed)[5] subspaces of the complete ranked spaces are not always complete.

Example 1. Let R be the set of all pairs $p=(x,y)$ of real numbers x, y. And let $E(n; f_1,\cdots,f_m)$, where m and n are positive integers and $f_i(1\leq i\leq m)$ is a straight line which passes over the origin $O=(0,0)$, denote the set of all points $p=(x,y)$ such that $x^2+y^2<\frac{1}{n^2}$ and $p\notin f_i$ for any i. Put $V(n; f_1,\cdots,f_m)=\{O\}\smile E(n; f_1,\cdots,f_m)$. The system of

---

1) See, for the notions and the terminologies, K. Kunugi: Sur les espaces complets et régulièrement complets. I, Proc. Japan Acad., **30**, 553–556 (1954); and H. Okano: Some operations on the ranked spaces. I, Proc. Japan Acad., **33**, 172–176 (1957).

2) Let the rank of R be defined by ω_ν.

3) $\tilde{\tilde{A}}\subseteq\tilde{A}$ is, in general, false. See, for example, H. Okano: *Op. cit.*, *Example 1.* Cf. C. Kuratowski: Topologie, I, 20 (1948).

4) 0 denotes the empty set.

5) A subset is called r-closed if it is closed for r-topology.

neighbourhoods of the origin is the family of all such $V(n; f_1, \cdots, f_m)$ and the neighbourhoods of another point are given by the translation. Then we have $\omega(R)=\omega_0$. So we can put $\mathfrak{B}_n$=the family of all neighbourhoods $V(n+1, f_1, \cdots, f_m)$ of all points. Then R is a complete ranked space. Now by S we shall denote the subspace of all points $p=(x, y)$ such that $y=0$ and $x \neq 0$. Then S is closed for the given topology but not complete.

Example 2. Let R be the same set as Example 1. And we shall give a topology as follows: for any positive integer n, V_n denotes the set of all points $p=(x, y)$ such that $x^2+y^2<\frac{1}{n^2}$ or such that $x>0$ and $y=0$. The neighbourhoods of the origin are $\{V_n\}$ and, for another point, neighbourhoods are given by the translation. Then $\omega(R)=\omega_0$. So we put $\mathfrak{B}_n$=the family of all neighbourhoods of type V_{n+1} of all points. Then R is complete. Denote by S the subspace of all points such that $y=0$. Then S is closed for the both topologies, that is $\bar{S}=\tilde{S}=S$, but S is not complete.

Now we shall give a sufficient condition for subspaces to be complete.

Lemma. Let R be complete and S be a subspace satisfying the conditions:

(1) If $p_\alpha(0 \leq \alpha < \gamma < \omega_\nu)$, $q \in S$, $\bigcap_\alpha U_\alpha(p_\alpha) \cap S \supseteq V(q) \cap S$, $\sup_\alpha \gamma_\alpha < \gamma'$ and $U_\alpha(p_\alpha) \in \mathfrak{B}_{\gamma_\alpha}$, $V(q) \in \mathfrak{B}_{\gamma'}$ in R, then there exists $W(q)$ such that $W(q) \in \mathfrak{B}_{\gamma''}$ in R, $\sup_\alpha \gamma_\alpha < \gamma'' \leq \gamma'$, $\bigcap_\alpha U_\alpha(p_\alpha) \supseteq W(q)$ and $W(q) \cap S \supseteq V(q) \cap S$.

(2) For any fundamental sequence $V=\{V_\alpha(p_\alpha)\}$ of R such that $p_\alpha \in S$, we have $\vartheta(V) \cap S \neq 0$.

Then S is complete.

Corollary. If R is complete and S is an r-closed subspace satisfying the condition (1) of the lemma, then S is complete.