149. On Boundary Values of Some Pseudo-Analytic Functions

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(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1957)

Let $\zeta=\varphi(z)$ be a quasi-conformal mapping from $|z|<1$ to $|\zeta|<1$. Then $\varphi(z)$ is not necessarily absolutely continuous function of $\arg z=t$ on $|z|=1$ (cf. [4]), although it is always continuous and of bounded variation for $0\leq t\leq 2\pi$. In the present short note we shall give a sufficient condition, for $\varphi(e^{it})$ to be absolutely continuous in t, in such a form, that it is applicable to generalization of some classical theorems. Our analysis is based essentially on Ahlfors's mapping theory [2, 3]. We must also remark that the result is closely related to one of the propositions stated in [5] without proof.

In this paper we use the following notations: for any complex number z, z^* is its inversion with respect to the unit circumference. Areal mean of a continuous function $g(z)$ over the disk $|z-a|\leq b$ shall be denoted by $M(g; a; b)$, i.e.

$$M(g;a;b)=\frac{1}{\pi b^2}\int_0^b\int_0^{2\pi} g(a+re^{it})\,rdtdr.$$

Any integral without explicit indication of its integration domain should be computed over the whole plane.

Lemma. *Given any function $g(z)$ in $|z|<1$ which fulfils the Hölder condition of order α*

$$|g(z_1)-g(z_2)|\leq A|z_1-z_2|^\alpha \quad |z_1|<1,\ |z_2|<1,\ 0<\alpha\leq 1,$$

then there exists a sequence of functions $\{g_n(z)\}$ in $|z|<\infty$, such as to satisfy the conditions

i) *$g_n(z)$ has a uniformly bounded carrier,*

ii) $|g_n(z_1)-g_n(z_2)|\leq B|z_1-z_2|^\alpha$,

iii) $\sup_{|z|<\infty}|g_n(z)|\leq \sup_{|z|<1}|g(z)|$, $\quad\begin{pmatrix}|z_1|<\infty,\ |z_2|<\infty\\ n=1,2,\cdots\end{pmatrix}$

iv) *$\{g_n(z)\}$ converges uniformly to $g(z)$ in $|z|<1$.*

Proof. For example we proceed as follows:

Set

$$\gamma(z)=\begin{cases} g(z^*) & 1<|z|\leq 4\\ 0 & |z|\geq 5.\end{cases}$$

And in the circular ring $4<|z|<5$, $\gamma(z)$ shall be equal to the solution of Dirichlet problem with the boundary values $g(z^*)$ on $|z|=4$, 0 on $|z|=5$.

Let $\delta>0$ be a sufficiently small number, with which we define the function

$$\gamma_0(z)=\begin{cases} g(z) & |z|<1 \\ \gamma(z) & 1<|z|\leq 2 \\ M(\gamma;\, z;\, \delta(|z|-2)) & 2\leq |z|\leq 3 \\ M(\gamma;\, z;\, \delta) & |z|\geq 3. \end{cases}$$

We see first, $\gamma_0(z)$ is locally Hölder-continuous with exponent α outside of the unit circle, and then globally as well, there. If $|z_1|<1<|z_2|<2$, $|\gamma_0(z_1)-\gamma_0(z_2)|=|\gamma_0(z_1)-\gamma_0(z_2^*)|\leq A|z_1-z_2^*|^\alpha\leq A|z_1-z_2|^\alpha$, since $|z|=1$ is Apollonius circle with respect to the pair z_2, z_2^*. So $|\gamma_0(z_1)-\gamma_0(z_2)|\leq$ const. $|z_1-z_2|^\alpha$ whenever $|z_1|\neq 1$, $|z_2|\neq 1$.

Put

$$g_n(z)=M(\gamma_0;\, z;\, \delta/n).$$

Then $\{g_n(z)\}$ is one of the desired sequences. Because, $\{g_n(z)\}$ possesses obviously the properties i), iii) and iv). As for ii), we have by definition

$$|\gamma_0(z_1+re^{it})-\gamma_0(z_2+re^{it})|\leq B|z_1-z_2|^\alpha$$

for almost all $r\in(0,\delta/n)$ and $t\in(0,2\pi)$. Q.E.D.

Theorem 1. *Let $\zeta=\varphi(z)$ be a continuously differentiable sense-preserving homeomorphism between the unit disks in $z(=x+iy=re^{it})$- and $\zeta(=\xi+i\eta=\rho e^{i\theta})$-plane respectively, which is conformal with respect to Riemannian metric $ds=|dz+h(z)d\bar{z}|$. Then, if $h(z)$ fulfils the Hölder condition of order α $(0<\alpha\leq 1)$, the boundary function $\theta=\theta(t)$ is absolutely continuous.*

Proof. We may assume $\varphi(0)=0$, $\varphi(1)=1$ without loss of generality. By Lemma we can choose a sequence of functions $\{h_n(z)\}$ converging uniformly to $h(z)$ in $|z|<1$, such that $|h_n(z_1)-h_n(z_2)|\leq B|z_1-z_2|^\alpha$ for any z_1 and z_2, $h_n(z)=0$ outside of some compact set, and that $|h_n(z)|\leq k<1$. Moreover, we may assume that the sequence $\{h_n(z)\}$ is uniformly convergent for $|z|<\infty$, since it forms a normal family there on account of the condition ii). Now, it is possible to construct the unique mapping $Z=\varphi_n(z)$ which is conformal in the metric $ds=|dz+h_n(z)d\bar{z}|$ and supplies a homeomorphism between the whole z- and Z-plane with the normalization

$$\varphi_n(0)=0,\quad \varphi_n(\infty)=\infty,\quad \lim_{z\to\infty}\frac{\varphi_n(z)}{z}=1$$

(cf. [2]). Then the sequence $\{\varphi_n(z)\}$ converges to a quasi-conformal mapping, $\zeta=\varphi_0(z)$ say, from $|z|\leq\infty$ to $|\zeta|\leq\infty$ (cf. [1]). Further it was proved in Ahlfors [3] that $\varphi_0(z)$ is conformal in $ds=|dz+h(z)d\bar{z}|$ almost everywhere in $|z|<1$ as follows: Soppose that $\varphi_0(z)$ is totally differentiable at $z=z_0$ $(|z_0|<1)$ and set

$$d\varphi_0(z_0)=p(z_0)dz+q(z_0)d\bar{z}.$$

A small square Q, with dimension d, centred at z_0 is, if fixed in a suitable direction, transformed by the locally affine mapping φ_n to a curvilinear quadrilateral Δ_n which is situated very near the rectangle of module $(|p(z_0)|+|q(z_0)|)/(|p(z_0)|-|q(z_0)|)$, so far as n is sufficiently large. One has

$$\frac{|p(z_0)|+|q(z_0)|}{|p(z_0)|-|q(z_0)|}<\mathrm{mod}\,\Delta_n+\varepsilon\leq\frac{1}{d^2}\iint_Q\frac{1+|h_n(z)|}{1-|h_n(z)|}dx\,dy+\varepsilon$$

by a slight modification of the module theorem. Let $n\to\infty$ and then $d\to 0$. Thus $|q(z)/p(z)|\leq|h(z)|$ almost everywhere in $|z|<1$. Changing the independent variable by a sense-preserving affine transformation $z=a\tau+b\bar{\tau}(|a|>|b|)$, one gets finally $q(z)/p(z)=h(z)$ a.e. in $|z|<1$.

Let D be the image of the unit disk by $Z=\varphi_0(z)$. We map D conformally onto the unit disk $|\zeta|<1$ by $\zeta=F(Z)$, so that $Z=0$, $\varphi_0(1)$ corresponds to $\zeta=0, 1$, respectively. Then we see that for $|z|<1$

$$\varphi(z)=F\circ\varphi_0(z). \qquad (*)$$

Let us denote $\psi_n(z)=\varphi_0(z)-z$. Then, for any rectifiable Jordan curve C, there holds the well-known Pompeiu's formula

$$\psi_n(z)=\frac{1}{2\pi i}\int_C\frac{\psi_n(w)}{w-z}dw+\frac{1}{\pi}\iint_{[C]}\frac{q_n(w)}{z-w}du\,dv \qquad w=u+iv,$$

where $[C]$ is the interior of C and $q_n(z)=\partial\psi_n(z)/\partial\bar{z}$. This can be brought to the form

$$\psi_n(z)=\frac{1}{\pi}\iint\frac{q_n(w)}{z-w}du\,dv$$

as $C\to\infty$, in virtue of the normalization. Since $q_n(z)$ is Hölder-continuous for $|z|<\infty$ (cf. [2]), we have

$$\frac{d\psi_n(e^{it})}{dt}=-i\left[e^{-it}q_n(e^{it})+\frac{e^{it}}{\pi}\iint\frac{q_n(w)}{(e^{it}-w)^2}du\,dv\right],$$

where the right-hand integral is to be taken as Cauchy principal value about e^{it}. We can extract from $\{q_n(z)\}$ a suitable subsequence $\{q_{n_\nu}(z)\}$ uniformly convergent everywhere. Since

$$\iint\frac{q_n(w)}{(w-e^{it})^2}du\,dv=\iint\frac{q_n(w)-q_n(e^{it})}{(w-e^{it})^2}du\,dv$$

and

$$\left|\frac{q_n(w)-q_n(e^{it})}{(w-e^{it})^2}\right|\leq m\,|w-e^{it}|^{\beta-2} \qquad (0<\beta<\alpha),$$

there exists by Lebesgue's theorem

$$\lim_{\nu\to\infty}\frac{d\psi_{n_\nu}(e^{it})}{dt}=-i\left[\frac{e^{it}}{\pi}\lim_{\nu\to\infty}\iint\frac{q_{n_\nu}(w)}{(w-e^{it})^2}du\,dv+e^{-it}\lim_{\nu\to\infty}q_{n_\nu}(e^{it})\right],$$

and $\{d\psi_{n_\nu}(e^{it})/dt\}$ is uniformly bounded for $0\leq t\leq 2\pi$. Therefore again by the same theorem

$$\varphi_0(e^{it_0})-\varphi_0(1)-e^{it_0}+1=\lim_{\nu\to\infty}[\psi_{n_\nu}(e^{it_0})-\psi_{n_\nu}(1)]=\int_0^{t_0}\left[\lim_{\nu\to\infty}\frac{d\psi_{n_\nu}(e^{it})}{dt}\right]dt,$$

which implies that $\varphi_0(e^{it})$ is absolutely continuous function in $t\in(0, 2\pi)$. Thus the boundary Γ of D is rectifiable Jordan curve with the representation

$$Z=Z(t) \qquad 0\leq t\leq 2\pi,$$

and this function transforms any set of linear measure zero on $|z|=1$ to a set of linear measure zero on Γ. Simple application of Riesz's theorem yields the absolute continuity of the function

$$\theta=\arg\varphi(e^{it})$$

in view of (∗). Q.E.D.

By *pseudo-analytic function* in a domain D in z-plane we imply the function $w=f(z)=u(x,y)+iv(x,y)$ satisfying there the conditions:

i) $f(z)$ is defined, one-valued and continuous;

ii) u_x, u_y, v_x, v_y exist and continuous;

iii) $J(z)=u_xv_y-u_yv_x>0$ with possible exception of at most the countable set S of points where $J(z)=0$, which accumulates nowhere inside of D;

iv) Dilatation of $f(z)$ is uniformly bounded for $z\notin S$.

It is evident that the function $h(z)=f_{\bar{z}}/f_z$ is defined and continuous for any point $z\notin S$. Here may be imposed on this eccentricity function $h(z)$ the further restriction (H):

(H) 1) for any point $z_0\in S$, $\lim_{z\to z_0} h(z)$ exists;
2) $h(z)$ is Hölder-continuous with some exponent α throughout D after the continuous prolongation 1).

Then we have an extension of Fatou's theorem:

Theorem 2. *Let $w=f(z)$ be a pseudo-analytic function in the unit disk $|z|<1$. If $f(z)$ is bounded and subject to the condition (H) in its definition domain, $f(z)$ possesses the well-determined limit values as z tends along Stolz paths to the periphery point e^{it} for every value of $t\in(0,2\pi)$ except possibly for a set of linear measure zero.*

Proof. Let R be the Riemann configuration generated by $w=f(z)$. It can be considered as the map by the analytic function $w=F(\zeta)$ for $|\zeta|<1(F(0)=f(0), F'(0)>0)$. Then $\frac{\partial\zeta}{\partial\bar{z}}\Big/\frac{\partial\zeta}{\partial z}=h(z)$. $F(\zeta)$ has an angular limit at every point of a set E of measure 2π on $|\zeta|=1$. By means of the quasi-conformal mapping $\zeta=\zeta(z)$, E corresponds to a set of measure 2π on $|z|=1$ on account of Theorem 1, while any Stolz path in $|z|<1$ is transformed to another such in $|\zeta|<1$ (cf. [6]). It follows that $f(z)$ has the property asserted. Q.E.D.

Some other theorems concerning the boundary correspondence by conformal mappings or the boundary values of analytic functions can be generalized in similar manners.

References

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