141. On Sobolev-Friedrichs' Generalisation of Derivatives

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We consider a fixed domain (open set) G in $R^n(x_1,\cdots,x_n)$ and in this note a function means always a complex valued measurable function defined on G and s is always a fixed non-negative integer. We identify two functions which coincide except on a null set. We use the following notations and definitions:

θ: the void set.

T_s: the set of finite sequences $(i_1, i_2,\cdots, i_p)$ of integers such that $1\leqq i_1, i_2,\cdots, i_p\leqq n$ $0\leqq p\leqq s$. The only sequence $(i_1,\cdots,i_p)$ for $p=0$ is the void set θ by definition.

D_i: the differentiation with respect to the variable x_i.

$$D_{(i_1,i_2,\cdots,i_p)}=D_{i_1}\cdot D_{i_2}\cdots D_{i_p} \quad (p\geqq 1).$$

$$D_\theta=I \text{ (the identity operator).}$$

$(f, g)^A=\int_A f\cdot\bar{g}dx_1\cdots dx_n$ for two functions f, g if $f\cdot\bar{g}$ is (Lebesgue) integrable on a measurable subset A of G.

$$\|f\|^A=\Big(\int_A |f|^2 dx_1\cdots dx_n\Big)^{1/2}.$$

We write (f, g), $\|f\|$ for $(f, g)^G$, $\|f\|^G$ respectively.

H: the set of functions such that $\|f\|<+\infty$.

$\mathfrak{H}$: the set of functions such that $\|f\|^A<+\infty$ for every compact set A contained in G.

C_s: the set of functions having continuous partial derivatives up to order s on G.

C_∞: the set of functions infinitely continuously differentiable on G.

Further following the authors above cited, we state some definitions and some propositions related to them for whose proofs we refer to K. O. Friedrichs [1, 2], Sobolev [6], L. Nirenberg [5]. If for a function U on G there are a set of functions $U_{(i_1,\cdots,i_p)}\varepsilon\mathfrak{H}$ $((i_1,\cdots,i_p)\varepsilon T_s)$ and a sequence of functions $f_m\varepsilon C_s$ $(m=1,2,\cdots)$ such that $U_\theta=U$ and $\|D_{(i_1,\cdots,i_p)}f_m-U_{(i_1,\cdots,i_p)}\|^A\to 0$ $(m\to\infty)$ for every compact set A contained in G and for every $(i_1,\cdots,i_p)\varepsilon T_s$, then U is said *strongly differentiable up to order s on G* and $U_{(i_1,\cdots,i_p)}$ are said *strong derivatives of U of order p*. We denote the set of functions strongly differentiable up to order s on G by $\mathfrak{H}_s$.

The strong derivative $U_{(i_1,\cdots,i_p)}$ of $U\varepsilon\mathfrak{H}_s$ is uniquely determined for each $(i_1,\cdots,i_p)\varepsilon T_s$.

$C_s\subset\mathfrak{H}_s$ and the strong derivative $U_{(i_1,\cdots,i_p)}$ of $U\varepsilon C_s$ is equal to

$D_{(i_1,\cdots,i_p)}U$ for each $(i_1,\cdots,i_p)\varepsilon T_s$.

Hence we write $D_{(i_1,\cdots,i_p)}U$ for strong derivatives $U_{(i_1,\cdots,i_p)}$ of any $U\varepsilon \mathfrak{H}_s$.

For $U, V\varepsilon \mathfrak{H}_s$ and a measurable subset A of G, we define $(U, V)_s^A$ by

$$(U, V)_s^A = \sum_{(i_1,\cdots,i_p)\in T_s} (D_{(i_1,\cdots,i_p)}U, D_{(i_1,\cdots,i_p)}V)^A$$

if $(D_{(i_1,\cdots,i_p)}U, D_{(i_1,\cdots,i_p)}V)^A$ have meanings for all $(i_1,\cdots,i_p)\varepsilon T_s$ and we define $\|U\|_s^A$ by

$$\|U\|_s^A=\{\sum_{(i_1,\cdots,i_p)\in T_s}(\|D_{(i_1,\cdots,i_p)}U\|^A)^2\}^{1/2}.$$

We write $(U,V)_s$, $\|U\|_s$ for $(U,V)_s^G$, $\|U\|_s^G$ respectively.

We denote the set of functions U such that $U\varepsilon\mathfrak{H}_s$ and $\|U\|_s<+\infty$ by E_s. E_s with the inner product $(U,V)_s$ becomes a Hilbert space. In the following, E_s is always endowed with this structure.

$E_s\cap C_\infty$ is the set of functions $U\varepsilon C_\infty$ with $D_{(i_1,\cdots,i_p)}U\varepsilon H$ for all $(i_1,\cdots,i_p)\varepsilon T_s$. We denote the closure in E_s of $E_s\cap C_\infty$, by H_s. H_s is a closed linear submanifold of E_s.

L. Nirenberg[1] proved $E_s=H_s$ for any bounded domain D with sufficiently smooth boundary.

In this note, we shall prove $E_s=H_s$ without any restriction on the domain D, following an idea of M. S. Narasimhan in another problem.[2]

§. The closure in G of the set of points where a continuous function on G does not vanish is said *carrier* of the function.

We define:

$\overset{\circ}{C}_\infty$: the set of functions εC_∞ whose carriers are compact.

$$\overset{\circ}{C}_\infty\subset E_s\cap C_\infty\subset H_s.$$

$\overset{\circ}{H}_s$: the closure of $\overset{\circ}{C}_\infty$ in E_s.

$\overset{\circ}{H}_s$ is a closed linear submanifold of E_s and $\overset{\circ}{H}_s\subset H_s$.

$(\overset{\circ}{H}_s)^\perp$: the orthogonal complement of $\overset{\circ}{H}_s$ in E_s.

We define differential operator Λ_s by

$$\Lambda_s=\sum_{p=0}^{s}(-\Delta)^p \quad \text{where } \Delta=\sum_{i=1}^{n}D_i^2.$$

Λ_s is a formally self-adjoint linear elliptic differential operator of order $2s$ with constant coefficients.

Lemma 1. If $U\varepsilon\mathfrak{H}_s$ and $\varphi\varepsilon\overset{\circ}{C}_\infty$, then

$$(U,\varphi)_s=(U,\Lambda_s\varphi).$$

Proof. Let compact A $(\subset G)$ be the carrier of φ. Since $U\varepsilon\mathfrak{H}_s$, there is a sequence of functions $f_m\varepsilon C_s$ $(m=1,2,\cdots)$ such that

$$\|U-f_m\|_s^A\to 0 \quad (m\to\infty). \tag{1}$$

1) Cf. L. Nirenberg [5].

2) Cf. M. S. Narasimhan [4].

For $f_m \varepsilon C_s$, we get easily by partial integrations

$$(f_m, \varphi)_s^A = (f_m, \varphi)_s = \sum_{(i_1,\dots,i_p)\in T_s} (D_{(i_1,\dots,i_p)} f_m, D_{(i_1,\dots,i_p)}\varphi)$$

$$\sum_{(i_1,\dots,i_p)\in T_s} (f_m, (-1)^p D^2_{(i_1,\dots,i_p)}\varphi) = (f_m, \sum_{p=0}^{s} (-\Delta)^p \varphi) = (f_m, \Lambda_s \varphi) = (f_m, \Lambda_s \varphi)^A$$

since $\varphi \varepsilon \overset{\circ}{C}_\infty$ and the compact A is the carrier of φ.

Letting $m \to \infty$ on both sides of the above equation, we get by (1)

$$(U, \varphi)_s^A = (U, \Lambda_s \varphi)^A$$

since $\|U\|_s^A$, $\|U\|^A$, $\|\varphi\|_s^A$, $\|\Lambda_s\varphi\|^A < +\infty$.

Hence

$$(U, \varphi)_s = (U, \varphi)_s^A = (U, \Lambda_s\varphi)^A = (U, \Lambda_s\varphi)$$

since A is the carrier of φ. Q.E.D.

Lemma 2. $U \varepsilon (\overset{\circ}{H}_s)^\perp$ if and only if $U \varepsilon E_s \cap C_\infty (\subset H_s)$ and $\Lambda_s U = 0$.

Proof. By the definitions of $\overset{\circ}{H}_s$ and $(\overset{\circ}{H}_s)^\perp$, $U \varepsilon (\overset{\circ}{H}_s)^\perp$ if and only if $U \varepsilon E_s$ and

$$(U, \varphi)_s = 0 \quad \text{for all } \varphi \varepsilon \overset{\circ}{C}_\infty.$$

Hence by Lemma 1, $U \varepsilon (\overset{\circ}{H}_s)^\perp$ if and only if

i) $U \varepsilon E_s$

and

ii) $U \varepsilon \mathfrak{H}$ and $(U, \Lambda_s\varphi) = 0$ for all $\varphi \in \overset{\circ}{C}_\infty$.

The condition ii) means that $U (\varepsilon \mathfrak{H})$ is a weak solution on G of the partial differential equation $\Lambda_s U = 0$, since Λ_s is a formally self-adjoint linear differential operator. But Λ_s is a linear elliptic differential operator with constant coefficients. Hence by the results of L. Schwartz and others,[3] condition ii) is equivalent to: $U \varepsilon C_\infty$ and U is a solution of $\Lambda_s U = 0$ in the ordinary sense. Therefore $U \in (\overset{\circ}{H}_s)^\perp$ if and only if $U \varepsilon E_s \cap C_\infty$ and $\Lambda_s U = 0$. Q.E.D.

Remark 1. If $U \varepsilon (\overset{\circ}{H}_s)^\perp$, U is even an analytic function on G, that is, a function whose real part and imaginary part are real analytic on each connected component of G, since Λ_s is a linear elliptic differential operator with constant coefficients.

Theorem.

$$E_s = H_s$$

for any domain G $(\subset R^n)$ and any non-negative integer s.

Proof.

$$\overset{\circ}{H}_s \subset H_s \subset E_s.$$

Also by Lemma 2, $(\overset{\circ}{H}_s)^\perp \subset H_s$.

But $E_s = (\overset{\circ}{H}_s)^\perp \oplus \overset{\circ}{H}_s$ ($\oplus$ means direct sum). Hence $E_s = H_s$.

Q.E.D.

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3) Cf. L. Schwårtz [7], L. Gårding [3], L. Nirenberg [5].

Remark 2. A function $\varepsilon \overset{\circ}{C}_s$ can be approximated in the sense of norm $\| \ \|_s$ by linear combinations of functions of the form

$$\prod_{i=1}^{n} H_{m_i}(x_i) \exp(-x_i^2/2)$$

where $H_m(x)$ $(m=0, 1, \cdots)$ are Hermite polynomials.[4] From this and Remark 1, it follows that the set of functions (εE_s) analytic on G in the sense as in Remark 1, is dense in E.

Remark 3. By a reasoning similar to one in Remark 2, using the result of M. S. Narasimhan [4], we can prove that the weak extension D_w in $L^2(G)$ of a linear elliptic differential operation D with analytic coefficients bounded on G, is the closure of its restriction which operates on analytic functions on G belonging to the domain of definition of D_w.

References

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4) This can be proved by an extension of the method used for the case $s=1$ by T. Kato on p. 200 of T. Kato [8], using Fourier transformation.