113. The Lebesgue Constants for (γ, r) Summation of Fourier Series

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(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1960)

1. The Euler method of summation associates with a given sequence $\{s_n\}$ the means

$$\sigma_{n,r}=\sigma_n=\sum_{\nu=0}^{n}\binom{n}{\nu}r^{\nu}(1-r)^{n-\nu}s_{\nu},\quad n=0,1,2,\cdots,$$

where r is a constant which satisfies $0<r\leq 1$. The case $r=1$ corresponds to the ordinary convergence. The Lebesgue constants for this method are given by L. Lorch [1] for the case $r=\frac{1}{2}$, i.e.

$$L\left(n;\frac{1}{2}\right)=\frac{2}{\pi^2}\log 2n+A+o(1)\qquad \text{as } n\to\infty,$$

where

$$A=-\frac{C}{\pi^2}+\frac{2}{\pi}\int_0^1\frac{\sin u}{u}du-\frac{2}{\pi}\int_0^{\infty}\left\{\frac{2}{\pi}-|\sin u|\right\}\frac{du}{u}\tag{1}$$

and C is the Euler-Mascheroni constant. For $0<r<1$ these constants are given by A. E. Livingston [2], i.e.

$$\begin{aligned}L(n;r)&=\frac{2}{\pi^2}\log\frac{2nr}{1-r}+A+o(1)\\&=L\left(nr/(1-r);\frac{1}{2}\right),\end{aligned}$$

where A is defined by (1).

Next the (γ, r) method of summation associates with a given sequence $\{s_n\}$ the means

$$\sigma^*_{n,r}=\sigma^*_n=\sum_{\nu=n}^{\infty}\binom{\nu}{n}r^{n+1}(1-r)^{\nu-n}s_{\nu},\quad n=0,1,2,\cdots,$$

where r is a constant which satisfies $0<r\leq 1$ [3]. Since the case $r=1$ corresponds to the ordinary convergence, we may suppose $0<r<1$. The object of the present note is to investigate the Lebesgue constants for (γ, r) method of Fourier series. We prove the following theorem.

Theorem. *The Lebesgue constants for (γ, r) method are given by*

$$\begin{aligned}L^*(n;r)&=\frac{2}{\pi^2}\log\frac{2n}{1-r}+A+o(1)\\&=L\left(\frac{n}{1-r};\frac{1}{2}\right)+o(1)\qquad \textit{as } n\to\infty,\end{aligned}$$

where A is defined by (1).

2. **Proof.** For the proof we are much indebted to A. E. Livingston [2]. We can easily see [4]

$$L^*(n; r)=\frac{2}{\pi}\int_0^{\pi}\left|\Im\left\{\frac{r^{n+1}e^{i\left(n+\frac{1}{2}\right)u}}{(1-e^{iu}+re^{iu})^{n+1}}\right\}\right|\frac{du}{2\sin\frac{u}{2}}$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\left|\Im\left\{\frac{r^{n+1}e^{i(2n+1)u}}{(1-e^{2iu}+re^{2iu})^{n+1}}\right\}\right|\frac{du}{\sin u}.$$

Here we put

$$\frac{1}{1-e^{2iu}+re^{2iu}}=p(u, r)e^{iq(u,r)}.$$

Then we get [4]

$$(2)\qquad \begin{cases} 1-\cos 2u+r\cos 2u=\dfrac{1}{p(u, r)}\cos q(u, r) \\ \sin 2u-r\sin 2u=\dfrac{1}{p(u, r)}\sin q(u, r) \end{cases}$$

$$(3)\qquad \frac{1}{p^2(u, r)}=r^2+2(1-r)(1-\cos 2u)$$

$$(4)\qquad 0\leq rp(u, r)\leq 1$$

where $rp(u, r)=1$ if and only if $u=0$. Thus

$$\Im\left\{\frac{r^{n+1}e^{i(2n+1)u}}{(1-e^{2iu}+re^{2iu})^{n+1}}\right\}=r^{n+1}p^{n+1}(u, r)\sin\{(n+1)q(u, r)+(2n+1)u\}$$

and

$$L^*(n; r)=\frac{2}{\pi}\int_0^{\pi/2}|\,r^{n+1}p^{n+1}(u, r)\sin\{(n+1)q(u, r)+(2n+1)u\}\,|\frac{du}{\sin u}.$$

Since

$$\int_0^{\pi/2}|\,r^{n+1}p^{n+1}(u, r)\sin\{(n+1)q(u, r)+(2n+1)u\}\,|\left\{\frac{1}{\sin u}-\frac{1}{u}\right\}du$$

$=o(1)$ as $n\to\infty$ from the Lebesgue principle of bounded convergence, we get

$$(5)\qquad L^*(n; r) =\frac{2}{\pi}\int_0^{\pi/2}|\,r^{n+1}p^{n+1}(u, r)\sin\{(n+1)q(u, r)+(2n+1)u\}\,|\frac{du}{u}+o(1).$$

The following two lemmas are important for estimating $L^*(n; r)$.

Lemma 1. For small $u\geq 0$ we get

$$q(u, r)=2su+O(u^3),$$

where $s=(1-r)/r$.

Proof. From (2)

$$\tan q(u, r)=\frac{(1-r)\sin 2u}{1-(1-r)\cos 2u}=\frac{(1-r)\sin 2u}{r+(1-r)(1-\cos 2u)}=\frac{s\sin 2u}{1+2s\sin^2 u}.$$

On the other hand for small u

$$\sin u=u+Au^3$$

$$\sin 2u = 2u + Au^3, \quad |A| \leq 2, \quad \text{and}$$

$$\frac{1}{1+x} = 1 + Bx, \quad |B| \leq 1, \quad \text{for } x \geq 0. \quad \text{Hence}$$

$$\frac{s \sin 2u}{1+2s\sin^2 u} = s(2u + Au^3)\{1 + 2Bs\sin^2 u\}$$
$$= 2su + O(u^3).$$

Next

$$\tan^{-1} x = x + Cx^3, \quad |C| \leq 4, \quad \text{for small } x \text{ and}$$

$$q(u, r) = \tan^{-1}\frac{(1-r)\sin 2u}{1-(1-r)\cos 2u} = 2su + O(u^3)$$

for $0 < r < 1$ and small u, so the lemma is proved.

The next lemma is a corollary of Livingston's lemma [2].

Lemma 2. If $1 < m < e$, then

$$r^2 p^2(u, r) < m^{-\frac{8(1-r)}{\pi^2 r^2}u^2}$$

for sufficiently small $u \geq 0$.

Proof. For $0 < y < 1$

$$\frac{1}{1+y} < 1 - \frac{y}{2} \leq m^{-\frac{y}{2}}, \quad \text{so that}$$

$$r^2 p^2(u, r) = \frac{1}{1 + \frac{4(1-r)}{r^2}\sin^2 u} \leq \frac{1}{1 + \frac{4(1-r)}{r^2}\left(\frac{2u}{\pi}\right)^2}$$
$$\leq m^{-\frac{1}{2}\frac{4(1-r)}{r^2}\left(\frac{2u}{\pi}\right)^2} = m^{-\frac{8}{\pi^2}\frac{(1-r)}{r^2}u^2}$$

for small u.

Next we see easily

$$L^*(n; r) = \frac{2}{\pi}\int_0^{\pi/2} |r^{n+1}p^{n+1}(u, r)\sin\{(n+1)q(u, r) + (2n+1)\}|\frac{du}{u} + o(1)$$
$$= \frac{2}{\pi}\left\{\int_0^{\varepsilon} + \int_{\varepsilon}^{\pi/2}\right\} + o(1) = \frac{2}{\pi}\int_0^{\varepsilon} + o(1)$$

as $n \to \infty$, from the Lebesgue principle of bounded convergence, where we may choose ε such that for u, $0 < u < \varepsilon$, Lemmas 1 and 2 are hold simultaneously. Next we put

$$I = \int_0^{\varepsilon} |r^{n+1}p^{n+1}(u, r)\sin\{(n+1)q(u, r) + (2n+1)u\}|\frac{du}{u}$$
$$- \int_0^{\varepsilon}\left|r^{n+1}p^{n+1}(u, r)\sin\left\{\frac{2nu}{r} + \frac{2-r}{r}u\right\}\right|\frac{du}{u}, \quad \text{then}$$

$$|I| \leq 2\int_0^{\varepsilon}\left|r^{n+1}p^{n+1}(u, r)\sin\frac{1}{2}\left\{(n+1)q(u, r) + (2n+1)u - \frac{2nu}{r} - \frac{2-r}{r}u\right\}\right.$$
$$\left.\cos\frac{1}{2}\left\{(n+1)q(u, r) + (2n+1)u + \frac{2nu}{r} + \frac{2-r}{r}u\right\}\right|\frac{du}{u}$$
$$\leq O(1)\int_0^{\varepsilon}(n+1)\, m^{-\frac{4}{\pi^2}\frac{(1-r)}{r^2}u^2(n+1)}u^2 du.$$

Since the integrand of the right member is bounded for any n and u, we get

$$\lim_{n\to\infty}\int_0^{\varepsilon}(n+1)m^{-\frac{4}{\pi^2}\frac{(1-r)}{r^2}u^2(n+1)}u^2du$$
$$=\int_0^{\varepsilon}\lim_{n\to\infty}\{(n+1)u^2m^{-\frac{4}{\pi^2}\frac{(1-r)}{r^2}u^2(n+1)}\}du=0$$

from the Lebesgue principle. Hence

$$L^*(n;r)=\frac{2}{\pi}\int_0^{\varepsilon}\left|r^{n+1}p^{n+1}(u,r)\sin\left\{\frac{2nu}{r}+\frac{2-r}{r}u\right\}\right|\frac{du}{u}+o(1)$$

and further

$$L^*(n;r)=\frac{2}{\pi}\int_0^{\varepsilon}\left|r^{n+1}p^{n+1}(u,r)\sin\frac{2nu}{r}\right|\frac{du}{u}+o(1)$$

from the same principle. Next

$$\int_0^{\varepsilon}\left|r^{n+1}p^{n+1}(u,r)\sin\frac{2nu}{r}\right|\frac{du}{u}$$
$$=\int_0^{\frac{r}{2(1-r)}n^{-\alpha}}+\int_{\frac{r}{2(1-r)}n^{-\alpha}}^{\varepsilon},\quad \text{where}\quad \frac{1}{4}<\alpha<\frac{1}{2}.$$

Here from Lemma 2

$$\int_{\frac{r}{2(1-r)}n^{-\alpha}}^{\varepsilon}\left(\frac{1}{1+\frac{4(1-r)}{r^2}\sin^2 u}\right)^{\frac{n+1}{2}}\left|\sin\frac{2nu}{r}\right|\frac{du}{u}$$
$$\leqq\int_{\frac{r}{2(1-r)}n^{-\alpha}}^{\varepsilon}m^{-\frac{4}{\pi^2}\frac{(1-r)}{r^2}u^2(n+1)}\frac{du}{u}$$
$$\leqq m^{-\frac{n^{1-2\alpha}}{\pi^2(1-r)}}\log\left(\frac{2\varepsilon(1-r)}{r}n^{\alpha}\right)=o(1)\quad \text{as } n\to\infty.$$

Hence

$$L^*(n;r)=\frac{2}{\pi}\int_0^{\frac{r}{2(1-r)}n^{-\alpha}}\left(\frac{1}{1+\frac{4(1-r)}{r^2}\sin^2 u}\right)^{\frac{n+1}{2}}\left|\sin\frac{2nu}{r}\right|\frac{du}{u}+o(1)$$
$$=\frac{2}{\pi}\int_0^{n^{-\alpha}}\left(\frac{1}{1+\frac{4(1-r)}{r^2}\sin^2\frac{ru}{2(1-r)}}\right)^{\frac{n+1}{2}}\left|\sin\frac{nu}{1-r}\right|\frac{du}{u}+o(1).$$

On the other hand

$$\log\frac{1}{1+\frac{4(1-r)}{r^2}\sin^2\frac{r}{2(1-r)}u}=\log r^2p^2\left(\frac{ru}{2(1-r)},r\right)$$
$$=-\frac{4(1-r)}{4r^2}\sin^2\frac{ru}{2(1-r)}+O(u^4)$$

$$=-\frac{u^2}{(1-r)}+O(u^4)=\frac{1}{1-r}(-u^2+O(u^4))$$
$$=\frac{1}{1-r}[\log\cos^2 u+O(n^{-4\alpha})] \qquad \text{for } 0<u<n^{-\alpha}.$$

Hence

$$r^{n+1}p^{n+1}\left(\frac{ru}{2(1-r)}, r\right)=(\cos u)^{\frac{n+1}{1-r}}\exp(O(n^{1-4\alpha}))$$
$$=(\cos u)^{\frac{n+1}{1-r}}(1+O(n^{1-4\alpha}))$$

and

$$L^*(n;r)=\frac{2}{\pi}\int_0^{n^{-\alpha}}(\cos u)^{\frac{n+1}{1-r}}\left|\sin\frac{nu}{1-r}\right|\frac{du}{u}$$
$$+O(n^{1-4\alpha})\int_0^{n^{-\alpha}}(\cos u)^{\frac{n+1}{1-r}}\left|\sin\frac{nu}{1-r}\right|\frac{du}{u}=J+K, \quad \text{say.}$$

Here

$$J=\frac{2}{\pi}\int_0^{n^{-\alpha}}(\cos u)^{\frac{n+1}{1-r}}\left|\sin\frac{nu}{1-r}\right|\frac{du}{u}$$
$$=\frac{2}{\pi}\int_0^{n^{-\alpha}}(\cos u)^{\frac{n}{1-r}}\left|\sin\frac{nu}{1-r}\right|\frac{du}{u}+o(1)$$

from the mean value theorem. On the other hand from the Lorch theorem

$$\frac{2}{\pi}\int_0^{n^{-\alpha}}(\cos u)^{\frac{n}{1-r}}\left|\sin\frac{nu}{1-r}\right|\frac{du}{u}=L\left(\frac{n}{1-r};\frac{1}{2}\right)+o(1)$$
$$=\frac{2}{\pi^2}\log\frac{2n}{1-r}+A+o(1) \quad \text{as } n\to\infty.$$

Finally $\qquad K=O(n^{1-4\alpha}\log n)=o(1) \quad \text{as } n\to\infty.$

Thus the theorem is proved.

References

[1] L. Lorch: The Lebesgue constants for $(E, 1)$ summation of Fourier series, Duke Math. Jour., **19** (1952).

[2] A. E. Livingston: The Lebesgue constants for (E, p) summation of Fourier series, Duke Math. Jour., **21** (1954).

[3] G. H. Hardy: Divergent Series (1949).

[4] K. Ishiguro: Gibbs' phenomenon for circle means (to be pressed).