126. Operator-Valued Entropy of a Quantum Mechanical Measurement*)

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The subject to be discussed is, from the point of view of Schrödinger theory, finding suitable measures for loss of definition of a state. From the point of view of hilbert space operator theory, it is simply finding suitable measures for the "extent of non-commutativity" of an operator D with the algebra generated by a discrete family of projections $\{E_i\}$. From the latter point of view, the interest of the paper is concentrated in §2.

In addition to the conspicuous lack of generality of the hypotheses, the statements proved here are incomplete in other respects. Suggestions for future research are accordingly made at several points.

1. States (Segal's approach) and entropy. In the usual Schrödinger theory, one may consider every state of a system to be the set of expectation values it assigns to observables: any state is a certain functional. The (vector) pure state corresponding to the wave-vector $\psi\in\mathcal{H}$ is the functional mapping every bounded hermitian A—'observable'—to the number $(A\psi, \psi)$; the mixed state corresponding to probabilities λ_i respectively of wave-vectors ψ_i ($\lambda_i\geqq 0, \sum\lambda_i=1$), maps A to $\sum\lambda_i(A\psi_i, \psi_i)$. The phases of ψ and ψ_i being irrelevant here, one may prefer to associate to the pure state, not the vector ψ, but instead the operator P of projection on the subspace $[\psi]\subset\mathcal{H}$; to the mixed state then, will be associated the operator $D=\sum\lambda_i P_i$, where P_i is the projection on $[\psi_i]$. The functional then takes A to $\operatorname{tr}(PA)$, respectively $\operatorname{tr}(DA)$. It is a simple and well-known, but important, fact that there is no loss of generality in assuming here that the P_i are orthogonal—D is after all simply an arbitrary positive (semi-) definite operator with trace 1.

Now I propose to seek a definition of entropy-increase which will measure the extent to which a state is made "more mixed" by being subjected to a measurement [11]. If the dials of the measuring instrument are read, the "packet is reduced", and entropy should decrease. If the dials are not read, the measurement replaces any

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pure state by a mixture of alternative pure states of unrelated phase, so some predictability from knowledge antedating the measurement is lost, and entropy should increase. The point of view taken here is that the quantity associated with a measurement, to preserve analogy to the entropy of a chance event, must refer to the second interpretation—the dials unread. Subsequent reading of the dials is analogous to determining the outcome of the event.

A measurement is characterized by a set of projections $E_j, \sum E_j = 1$. It replaces any ψ by $\chi_j = E_j\psi / \| E_j\psi \|$, with probability $\| E_j\psi \|^2$. Let us consider, instead of the functional of evaluating every A in the state ψ, the following functional: one first subjects ψ to the measurement $\{E_j\}$ and then evaluates the observable A. This functional maps A to the number $\sum \| E_j\psi \|^2 (A\chi_j, \chi_j) = \sum (AE_j\psi, E_j\psi) = \operatorname{tr}(A\sum E_j D E_j)$. This means that, independent of A, the measurement has replaced D by $\sum E_j D E_j$: its effect is a 'pinching' [4] on the state operators. The first desideratum for a measure of entropy is, then, that $H(\sum E_j D E_j) \geqq H(D)$ always hold. Call this *condition* A.

But in addition it may happen that one decides to ignore available information by averaging over some variable. If D commutes with the corresponding observable, this corresponds to replacing a direct sum $D_1 \oplus D_2 \oplus \cdots \oplus D_k$ (which is a direct summand of D) by the direct sum of k copies of $k^{-1}(D_1 + D_2 + \cdots + D_k)$, because D enters linearly in the expression $\operatorname{tr}(DA)$ for the value of the functional. This makes sense provided the k subspaces are isomorphic in a natural way: i.e., differ only in the eigenvalue of the variable being averaged over. But then it is natural to say that what one has done is to average (over all permutations) the operators that differ from D only in the permutation of these subspaces. And it is natural then to require of the entropy that it be if anything increased by the process: $H(D)$ must be a concave function of D. Call this *condition* B.

Finally, if some sense can be given to the equation $H(D_1 \otimes D_2) = H(D_1) + H(D_2)$, it should preferably also hold (*condition* C). Here D_1 and D_2 are states of different systems, and $D_1 \otimes D_2$ is a state of the combined system such that no measurement involving variables of only one subsystem can give information about the other. The motivation for desiring this equation to hold is thereby plain.

2. Numerical-valued and operator-valued entropy. I. E. Segal [10] proposed the definition $\operatorname{tr} h(D)$ for the entropy; here and in the rest of the paper, h denotes the real continuous function on $[0, 1]$ defined by $h(t) = -t \log t$ $(t > 0)$. (This may fail to be finite even though $\operatorname{tr} D = 1$, as simple examples show, but this is a relatively insignificant drawback.) Segal proved that this entropy possesses the property B of concavity [10, Theorem 1] and a property generalizing condition A [10, Theorem 2]. It also satisfies condition C, as one verifies immediately.

What can we say about the uniqueness of h in Segal's definition? That is, if entropy is measured by $\mathrm{tr}\, f$, for real function f, how do the conditions of the previous section restrict f? Segal points out that $\mathrm{tr}\, f$ is a concave function of operator argument if f is merely any concave real function on $[0, 1]$. I will give a proof, since it is very simple, and neither Segal nor von Neumann whom he cites [11] published any. Any two bounded hermitian operators possessing trace can be approximated uniformly by finite matrices, the ranges of which are contained in a common larger finite-dimensional subspace of $\mathcal{H}$, so it is enough to prove concavity for $n \times n$ matrix arguments. Now $\mathrm{tr}\, f(A)$, for such A, may be regarded as a real function of n real variables: $\mathrm{tr}\, f(A) = \sum f(a_\nu)$, the a_ν being the eigenvalues of A. As such, it is evidently concave. But this is known [3] to imply that it is concave as a function of a matrix argument.

Condition A is known to follow from condition B, cf. [4, §6], so will also be satisfied by functions from the wider class. This is all that is required for the following:

THEOREM 1. *If f is any concave real function, then* $\mathrm{tr}\, f$ *satisfies conditions* A *and* B *of* §1.

Presumably, to impose the further condition C would essentially characterize the function h. I do not know of any proof of this. Compare [6], [7].

Nakamura and Umegaki [8] have pointed out that the first two properties of the entropy proved by Segal are consequences of the two theorems which follow.

THEOREM 2. *h is an operator-concave function on* $(0, \infty)$.

That is, for any self-adjoint A, B which spectrum $\subset (0, \infty)$, and real $\lambda \in (0, 1)$,

$$h((1-\lambda)A + \lambda B) \geqq (1-\lambda)h(A) + \lambda h(B). \tag{1}$$

This is easy using the criterion of Bendat and Sherman [1, Thm. 3.2], which may also be found, in a somewhat novel presentation, in a forthcoming paper [5]: A function f is operator-convex in an interval if and only if its first divided difference (=difference quotient) is operator-monotone there with respect to either variable separately. So for $s, t \in (0, \infty)$ one must show that $-h^{[1]}(s,t) = \dfrac{s \log s - t \log t}{s-t}$ is operator-monotone as a function of t. But this follows from Loewner's criterion [1], for it can clearly be extended analytically from its interval of definition to the entire upper half plane, and then maps the latter into itself.

To be sure, the operators D which are in question always have 0 in their spectrum. However, this difficulty is easy to circumvent also, for example by replacing A by $A+\varepsilon\geqq\varepsilon>0$ in (1), and B similarly; when $\varepsilon\to 0$, each term converges uniformly to its counterpart in (1), so the inequality is preserved.

I suggest that it would be of interest to find a direct elementary proof of Theorem 2, not using the general Loewner theory; cf. [5, §8].

THEOREM 3. *$-h$ satisfies the Sherman condition.*

The Sherman condition [2] for a real function f may be equivalently phrased this way: for any pinching $\{E_i\}$, and any D as before, $f(\sum E_iDE_i)\leqq\sum E_if(D)E_i$; see [9], [5]. It is known [2] to be equivalent to operator-convexity.

Since, by Theorem 1, $\operatorname{tr} f$ satisfies conditions A and B for much more general f than the particular function $f=h$, Segal's theorems seem a rather small reward to extract from these two theorems about h. I propose to broaden the viewpoint a little, eliminating the role of the trace, in a way that seems consistent with §1; see also comment (4) of [10].

We know from Theorem 3 that $h(\sum_i E_iDE_i)\geqq\sum_i E_ih(D)E_i$ holds identically. Accordingly, if we adopt the operator

$$H=H(\{E_i\},D)=h(\sum E_iDE_i)-\sum E_ih(D)E_i$$

as the measure of the entropy of the measurement $\{E_i\}$ with respect to the state D, we have the general property that $H(\{E_i\},D)\geqq 0$.

H is a "measure of loss of definition" as referred to in the introduction and §1, and is the "operator-valued entropy" of the title.

This has used, of course, only the operator-concavity of h. Indeed the following stronger statement holds:

THEOREM 4. *Let f be any function operator-concave on $(0,\infty)$, but not linear. Let $F=f(\sum E_iDE_i)-\sum E_if(D)E_i$ (D, E_i as before). Then $F\geqq 0$, and the nullspace $\mathfrak{N}$ of F is the maximal subspace of $\mathcal{H}$ on which the restrictions of D and the E_i commute.*

The part still awaiting proof is the last assertion. Any subspace invariant under D, E_i, on which the restrictions of D and the E_i commute, has the same property with D replaced by $f(D)$; so $F=f(D)-f(D)=0$; this for general f.

In the other direction, prove first for the case $f(t)=-t^2$. Let E be one of the E_i such that $ED-DE\neq 0$ and $FE=0$, if possible. By definition, $FE=ED(1-E)DE=A^*A$, with $A=(1-E)DE$. Supposing x such that $(ED-DE)x\neq 0$, there is no loss in supposing $x=Ex$, when this becomes $0\neq(EDE-DE)x=-Ax$, hence $0\neq A^*Ax=FEx$, a contradiction.

A similar elementary argument proves the result for the special case $f(t)=-t^{-1}$; changing the independent variable as required, the result is proved for $f(t)=(\tau-t)^{-1}$, $\tau<0$. For linear f, of course, $F\equiv 0$.

Any operator-concave function f can be expressed [1] as a sum of positive multiples of the types so far considered (with a Stieltjes integral with respect to τ). This leads to a corresponding expression of (Fx, x) ($x\in\mathcal{H}$) as a sum of terms each of which is $\geqq 0$, all of which therefore must be 0 for $x\in\mathfrak{N}$. But if any of these constituents is non-linear, this implies $(ED-DE)x=0$.

This theorem shows that any of these F is a good "measure of the extent to which D fails to commute with the E_i". For a numerical measure, any $\sigma\circ F$, with σ a positive linear functional on the hermitian operators, is suitable. Finally, by Theorem 1, the choice of the trace for σ allows wider choice of f, which then need be merely concave in the usual sense.

(I remark that I. Schur used, as a measure of the failure of a matrix $C=((c_{ij}))$ to be diagonal, the quotient $\det C/\Pi_i c_{ii}$, which is 1 if and only if C is diagonal, otherwise smaller. The negative of the logarithm of this may be written as $\operatorname{tr}\log(\sum E_iCE_i)-\operatorname{tr}\log C$, so fits into the class described in Theorem 1, namely by choosing $f(t)=\log t$; but the logarithm function, though concave on $(0, \infty)$, is not operator-concave on any interval, so it would not be possible to use $\sigma(\log(\sum E_iCE_i))-\sigma\sum E_i(\log C)E_i$ in the same way for general σ.)

Again, it would be of interest to characterize h among operator-concave f by making further natural requirements on $\sigma\circ f$ beside positivity of $\sigma\circ F$. I do not know of any such characterization.

References

[1] J. Bendat and S. Sherman: Monotone and convex operator functions, Trans. Amer. Math. Soc., **79**, 58–71 (1955).

[2] C. Davis: A Schwarz inequality for convex operator functions, Proc. Amer. Math. Soc., **8**, 42–44 (1957).

[3] ——: All convex invariant functions of hermitian matrices, Arch. Math., **8**, 276–278 (1957).

[4] ——: Various averaging operations onto subalgebras, Illinois J. Math., **3**, 538–553 (1959).

[5] ——: Notions generalizing convexity for functions defined on spaces of matrices, American Mathematical Society, Proceedings of Symposia in Pure Mathematics, Providence, vol. **5**: Convexity, (1962).

[6] A. Ya. Hinčin: The concept of entropy in the theory of probability, Uspehi Mat. Nauk (N.S.) (Russian) **8**, no. 3(55), 3–20 (1953). Translation: American Mathematical Society Translations, Providence, series 2, vol. **12**, 181–197 (1959).

[7] A. Rényi: Dimension, entropy, and information, Transactions of the Second Prague Conference on Information Theory, Statistical Decision functions, Prague, Random Processes, 545–556 (1960).

[8] M. Nakamura and H. Umegaki: A note on the entropy for operator algebras, Proc. Japan Acad. **37**, 149–154 (1961).

[9] M. Nakamura, M. Takesaki, and H. Umegaki: A remark on the expectations of operator algebras, Kōdai Math. Sem. Rep. **12**, 82–90 (1960).
[10] I. E. Segal: A note on the concept of entropy, J. Math. Mech. **9**, 623–629 (1960).
[11] J. von Neumann: Mathematische Grundlagen der Quantenmechanik, Berlin, Springer (1932).