39. On Neutral Elements in Lattices

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1. Introduction. An element n of a lattice L is called neutral if and only if $\{n, x, y\}$ generates a distributive sublattice of L for any pair of elements x, y of L. It has been studied by many authors to define neutral elements by some equalities. For instance, Grätzer and Schmidt [2] have shown

LEMMA 1. *An element n of L satisfying the following equalities is neutral:*

$$x\smile(n\frown y)=(x\smile n)\frown(x\smile y),\quad x\frown(n\smile y)=(x\frown n)\smile(x\frown y)$$

for all $x, y\in L$.

And they have proposed the question whether the neutrality can be defined by a single equality or not. In the present paper we intend to answer this question by proving

THEOREM 1. *An element n of a lattice L is neutral if it satisfies*

$$(n\frown x)\smile(n\frown y)\smile(x\frown y)=(n\smile x)\frown(n\smile y)\frown(x\smile y)$$

for all $x, y\in L$.

Again it is well-known [1] that an element n of a complemented modular lattice is neutral if and only if its complement is unique. Grätzer and Schmidt [2] have stated a generalized theorem in modular lattices and proposed to generalize this fact for relatively complemented lattices. In response to the proposal we shall show

THEOREM 2. *Let L be a relatively complemented lattice with 0 and 1. An element n of L is neutral if and only if it has a unique complement.*

2. Proof of Theorem 1. Suppose that $n\in L$ satisfies

$$(n\frown x)\smile(n\frown y)\smile(x\frown y)=(n\smile x)\frown(n\smile y)\frown(x\smile y) \tag{1}$$

for all $x, y\in L$ and put $a=x\frown(n\smile y), b=y\frown(n\smile x)$.

Substituting $n\smile y$ for y in (1), we have $n\smile(x\frown(n\smile y))=(n\smile x)\frown(n\smile y)$; namely $n\smile a=(n\smile x)\frown(n\smile y)\geqq b$ and $n\smile a\geqq a\smile b$; similarly $n\smile b\geqq a\smile b$. Then using (1) with respect to n, a, b, we get

$$\begin{aligned}a\leqq a\smile b&=(n\smile a)\frown(n\smile b)\frown(a\smile b)=(n\frown a)\smile(n\frown b)\smile(a\frown b)\\&=(n\frown x)\smile(n\frown y)\smile(x\frown y)\leqq n\smile(x\frown y).\end{aligned}$$

Now put $c=(x\frown n)\smile(x\frown y)$. Then $n\frown a\leqq c\leqq a$. Substituting a, c for x, y in (1), we have $(n\frown a)\smile c=(n\smile c)\frown a$, whence $c=(n\frown a)\smile c=(n\smile c)\frown a=(n\smile(x\frown y))\frown a=a$.

Thus $x\frown(n\smile y)=(x\frown n)\smile(x\frown y)$, and dually $x\smile(n\frown y)=(x\smile n)\frown(x\smile y)$. So it follows from Lemma 1 that n is neutral.

3. Proof of Theorem 2. Let L be a relatively complemented lattice and n an element of L which has only one relative complement in any interval containing it. Given $x, y \in L$, we put $a = x \smile (n \frown y)$ and we shall first show $n \frown a = (n \frown x) \smile (n \frown y)$.

Evidently $n \frown x \leqq (n \frown x) \smile (n \frown y) \leqq n \frown a \leqq a$. Let u be a relative complement of $n \frown a$ in the interval $[(n \frown x) \smile (n \frown y), a]$ and v a relative complement of $(n \frown x) \smile (n \frown y)$ in $[n \frown x, u]$. Then we have

$$n \smile v = n \smile (n \frown x) \smile (n \frown y) \smile v = n \smile u = n \smile (n \frown a) \smile u = n \smile a = n \smile x$$

and $$n \frown v = n \frown a \frown u \frown v = ((n \frown x) \smile (n \frown y)) \frown v = n \frown x.$$

So v is the relative complement of n in $[n \frown x, n \smile x]$ and hence $v = x$. Thus we have $u = v \smile (n \frown x) \smile (n \frown y) = a$ and $(n \frown x) \smile (n \frown y) = u \frown (n \frown a) = n \frown a$.

Now let s be a relative complement of $b = (x \smile n) \frown (x \smile y)$ in the interval $[a, x \smile y]$. Since $n \smile b \geqq n \smile a = n \smile x \geqq n \smile b$, we have

$$n \smile s = n \smile a \smile s = n \smile b \smile s = n \smile x \smile y$$

and $n \frown s = n \frown (x \smile n) \frown (x \smile y) \frown s = n \frown a = (n \frown x) \smile (n \frown y)$.

Again if t is a relative complement of $(y \smile n) \frown (y \smile x)$ in the interval $[y \smile (n \frown x), x \smile y]$, then interchanging x and y in the above statement, we have $n \smile t = n \smile x \smile y$ and $n \frown t = (n \frown x) \smile (n \frown y)$. So both s and t are relative complements of n in the interval $[(n \frown x) \smile (n \frown y), n \smile x \smile y]$ and from the first assumption we get $s = t$, whence $s \geqq y \smile (n \frown x) \geqq y$, $s \geqq x \smile y \geqq b$ and $a = s \frown b = b$. Thus $x \smile (n \frown y) = (x \smile n) \frown (x \smile y)$ and dually $x \frown (n \smile y) = (x \frown n) \smile (x \frown y)$. So n is neutral and we infer

LEMMA 2. *If an element n of a relatively complemented lattice L has only one relative complement in any interval containing it, then n is neutral.*

Now let L be a relatively complemented lattice with $0, 1$ and n an element of L having only one complement n' in L. We shall show that the relative complement x of n in an interval $[a, b]$, where $a \leqq n \leqq b$, is uniquely determined. If y is a relative complement of a in $[0, x]$ and z a relative complement of b in $[y, 1]$, then we get

$$n \frown z = n \frown b \frown z = n \frown y = n \frown x \frown y = a \frown y = 0$$

and $$n \smile z = n \smile a \smile y \smile z = n \smile x \smile z = b \smile z = 1.$$

Namely z is the complement of n and hence coincides with n'. So $x = (z \frown b) \smile a = (n' \frown b) \smile a$ is uniquely determined in $[a, b]$. Therefore Theorem 2 mentioned at the beginning is immediately deduced from Lemma 2.

References

[1] G. Birkhoff: Lattice Theory, Revised ed., Amer. Math. Soc. Coll. Publ., Vol. 25, New York (1948).

[2] G. Grätzer and E. T. Schmidt: Standard ideals in lattices, Acta Math. Acad. Sci. Hung., **12** (1961).