65. On Regular Algebraic Systems

A Note on Notes by Iseki, Kovacs, and Lajos

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L. Kovacs [2], K. Iseki [1], and S. Lajos [3] characterized regular rings and semigroups as algebraic systems satisfying the property $R\cap L=RL$ for any right ideal R and any left ideal L. A semigroup $(S,\cdot)$ and a ring or semiring $(S,+,\cdot)$ is regular iff for each $s\in S$ there exists an $x\in S$ such that $sxs=s$. Clearly, this follows from the statement: for each $s\in S$, there exist $x, y\in S$ such that $sxys=s$. The two statements are equivalent, for, if for each $s\in S$ there exists an $x\in S$ such that $sxs=s$, then also there exist a $z\in S$ such that $x=xzx=x(zx)=xy$ and therefore $sxys=s$.

In this communication we shall give a unified generalization of the characterizations of Kovacs, Iseki, and Lajos. It turns out that the description of regularity in terms of ideals is intrinsic to associative operations in general.

By an *algebraic system* $(A, o_1,\cdots, o_n)$ or simply A is meant a set A closed under a collection of m_i-ary operations o_i and often also satisfying a fixed set of laws. For instance, an m-ary operation $(\cdots)$ on A satisfies the *associative law* iff for each $x_1,\cdots,x_{2m-1}\in A$, $((x_1x_2\cdots x_m)x_{m-1}\cdots x_{2m-1})=(x_1(x_2x_3\cdots x_{m-1})\cdots x_{2m-1})=\cdots=(x_1x_2\cdots(x_{m-1}x_{m-2}\cdots x_{2m-1}))$. A is said to be *regular* with respect to the operation $(\cdots)$ iff for each $a\in A$ there exist $x_2, x_3,\cdots, x_m; y_1\, y_3,\cdots, y_m;\cdots; z_1, z_2,\cdots, z_{m-1}\in A$ such that

$$((ax_2\cdots x_m)(y_1ay_3\cdots y_m)\cdots(z_1z_2\cdots z_{m-1}a))=a.$$

Note that if A is both associative and regular relative to the operation, then the preceding identity may be rewritten as follows:

$$((ax_2\cdots x_m)(y_1a\cdots y_m)\cdots(z_1z_2\cdots a))=(a(x_2\cdots x_my_1)a\cdots(z_1z_2\cdots z_{m-1})a))$$
$$=(av_1a\cdots(\cdots v_{m-1}a))=a \text{ for some } v_1,\cdots,v_{m-1}\in A.$$

A subset S of A constitues a *subsystem* iff S is closed under the same operations and satisfies the same fixed laws in A.

Following G. B. Preston [4], a *j-ideal* $j=1,\cdots,m$ relative to the m-ary operation $(\cdots)$ is defined to be a subsystem I_j such that for any $x_1, x_2,\cdots, x_m\in A$, if $x_j\in I_j$ then $(x_1x_2\cdots x_m)\in I_j$. The j-ideal relative to $(\cdots)$ generated by an element $a\in A$ (usually called a principal j-ideal) is denoted by

$$(a)_j=(AA\cdots\overset{j}{a}\cdots A)\cup\{a\}.$$

A subsystem I which is a j-ideal for each $j=1,\cdots,m$ is simply called an *ideal.*

THEOREM 1. *In any algebraic system which is associative relative to an m-ary operation* $(\cdots)$, *the following conditions are equivalent*:

(1) *A is regular relative to the operation* $(\cdots)$;

(2) $(I_1 I_2 \cdots I_m)=\bigcap_{j=1}^{m} I_j$ *for any set of j-ideals* I_j *relative to the operation*;

(3) $((a_1)_1(a_2)_2\cdots(a_m)_m)=\bigcap_{j=1}^{m}(a_j)_j$ *for any set of elements* $a_1, a_2,\cdots, a_m \in A$;

(4) $((a)_1(a)_2\cdots(a)_m)=\bigcap_{j=1}^{m}(a)_j$ *for each element* $a\in A$.

Proof. To prove (1) implies (2) let A be regular relative to the m-ary operation $(\cdots)$ and let $a\in\bigcap_{j=1}^{m} I_j$ for any set of m j-ideals I_j relative to the operation. Then by regularity there exists $x_2,\cdots, x_m; y_1, y_3,\cdots, y_m,\cdots; z_1,\cdots, z_{m-1}\in A$ such that

$$((ax_2\cdots x_m)(y_1 a\cdots y_m)\cdots(z_1\cdots z_{m-1}a))=a.$$

I_j being a j-ideal for each $j=1,\cdots,m$, we thus obtain $(ax_2\cdots x_m)\in I_1$, $(y_1 a\cdots y_m)\in I_2,\cdots$, and $(z_1\cdots z_{m-1}a)\in I_m$ and hence $\bigcap_{j=1}^{m} I_j \subseteqq (I_1 I_2\cdots I_m)$. Conversely, if $a\in(I_1 I_2\cdots I_m)$ then $a=(i_1 i_2\cdots i_m)$ for $i_j\in I_j, j=1,\cdots,m$, and therefore $a\in I_j$ for each $j=1,\cdots,m$. Whence (2) is proved.

(2) implies (3) implies (4) are obvious.

Now to prove (4) implies (1) suppose $((a)_1(a)_2\cdots(a)_n)=\bigcap_{j=1}^{m}(a)_j$ for each $a\in A$. Since for each $a\in A, a\in\bigcap_{j=1}^{m}(a)_j$, then $a(b_1 b_2\cdots b_m)$ where either $b_k=a$ or $b_k=(c_1 c_2\cdots c_m)$ with $c_k=a$. Replace any one of the b_k's such that $b_k=a$ by its equal $a=(b_1 b_2\cdots b_m)$. Thus we can write a in any case in the form

$$a=(b_1 b_2\cdots b_m)=((ax_2\cdots x_m)(y_1 a\cdots y_m)\cdots(z_1 z_2\cdots z_{m-1}a))$$

for some $x_2,\cdots, x_m; y_1, y_3,\cdots, y_m;\cdots; z_1,\cdots, z_{m-1}\in A$. This shows that A is regular with respect to the operation.

The m-ary operation $(\cdots)$ will be called *commutative* iff for each $x_1,\cdots, x_m\in A$ and for each permutation ϕ of the integers $1,\cdots,m$

$$(x_1 x_2\cdots x_m)=(x_{\phi(1)}x_{\phi(2)}\cdots x_{\phi(m)}).$$

THEOREM 2. *An algebraic system A which is associative and commutative relative to an m-ary operation* $(\cdots)$ *is regular with respect to the same operation iff every ideal I of A is idempotent*, i.e. $(II\cdots I)=I$.

Proof. If A is commutative relative to $(\cdots)$, then $(aA\cdots A)$ $=(Aa\cdots A)=\cdots=(AA\cdots a)$ and hence every j-ideal is also a k-ideal for all $j, k=1,\cdots,m$. Hence by regularity

$(II\cdots I)=I\cap I\cap\cdots\cap I=I$ for each ideal I in A.

Conversely, suppose every ideal in A is idempotent. If $I_1, I_2,\cdots, I_m$ is any collection of ideals in A, then $\bigcap_{j=1}^{m} I_j$ is also an ideal and therefore

$$\bigcap_{j=1}^{m} I_j=(\bigcap_{j=1}^{m} I_j \bigcap_{j=1}^{m} I_j\cdots \bigcap_{j=1}^{m} I_j)\subseteqq(I_1I_2\cdots I_m),$$

inasmuch as I_j contains the intersection for each j. Furthermore, since each I_j, $j=1,\cdots,m$ is also a j-ideal, then $(I_1I_2\cdots I_m)\subseteqq\bigcap_{j=1}^{m} I_j$. Whence the conclusion follows.

Note that in case $(\cdots)$ is an associative m-ary operation in A, one may conveniently abbreviate $(aa\cdots a)=a^m$, $(a^m a\cdots a)=a^{2m-1}$, $(a^m a^m \cdots a)=a^{3m-2}$, $\cdots$, $(a^m a^m\cdots a^m)=a^{m^2}=a^{(m+1)m-m}$. Thus the admissible exponents of compositions of rank at most 2 are each of the form $km-k+1$ for some integer. Proceeding inductively, suppose that k_1m-k_1+1, $k_2m-k_2+1,\cdots$, k_mm-k_m+1 are previously known admissible exponents, then the exponent

$$\sum_{i=1}^{m}(k_im-k_i+1)=(\sum_{i=1}^{m}k_i-1)m-\sum_{i=1}^{m}k_i$$

of $(a^{k_1m-k_1+1}a^{k_2m-k_2+1}\cdots a^{k_mm-k_m+1})$ is evidently also of the same form. Hence every admissible exponent of an m-ary operation is of the form $km-k+1$.

An element $0\in A$ such that $(0x_1\cdots x_{m-1})=(x_1 0\cdots x_{m-1})=(x_1\cdots x_{m-0})=0$ for all $x_1,\cdots,x_{m-1}\in A$ is called *zero*. A *nilpotent* element $a\in A$ is one which satisfies $a^{km-k+1}=0$ for some integer k greater than 0.

THEOREM 3. *An algebraic system A which is commutative and associative and has a 0 with respect to an m-ary operation $(\cdots)$ possesses no nilpotent element other than* 0.

Proof. For each $0\neq a\in A$, let $[a]$ denote the subsystem of A generated by a, which may be inductively defined as follows:

(a) $a\in[a]$;

(b) $a^m\in[a]$;

(c) whenever $a^{n_1},\cdots,a^{n_m}\in[a]$, then also $a^{n_1+\cdots+n_m}\in[a]$.

To prove the theorem it suffices to show that $0\notin[a]$. We proceed inductively.

(a) $a\neq 0$ by assumption;

(b) $a^m\neq 0$. For, if $a^m=0$, then by virtue of the associativity, commutativity, and regularity of the given operation, there exists $x_1,\cdots,x_{m-1}\in A$ such that $a=(ax_1a\cdots(\cdots x_{m-1}a))=((aa\cdots a)x_1\cdots x_{m-1})=(a^m x_1\cdots x_{m-1})=(0x_1\cdots x_{m-1})=0$ contrary to (a).

(c) We now show that if $a^{n_1},\cdots,a^{n_m}$ are all non-zero elements of $[a]$ then $(a^{n_1}a^{n_2}\cdots a^{n_m})=a^{m_1+n_2+\cdots+n_m}\neq 0$. Suppose $a^{n_1n_2+\cdots+n_m}=0$.

Then by a remark above, we have

$$n_i = k_i m - k_i + 1 \quad \text{for } i = 1, 2, \cdots, m.$$

Since $(\cdots)$ is commutative, it may be assumed without loss of generality that $n_1 = \max_i n_i$. Then

$$\begin{aligned} mn_1 &= \sum_{i=1}^{m} n_i + (mn_1 - \sum_{i=1}^{m} n_i) = \sum_{i=1}^{m} n_i + \sum_{i=1}^{m} (n_i - n_i) = \sum_{i=1}^{m} n_i + \sum_{i=1}^{m} [(k_1 - k_i)m \\ &\quad - (k_1 - k_i)] = \sum_{i=1}^{m} n_i + (\sum_{i=1}^{m} (k_1 - k_i)m - \sum_{i=1}^{m} (k_1 - k_i) - m + 2) + (m-2) \\ &= \sum_{i=1}^{m} n_i + \{[\sum_{i=1}^{m} (k_1 - k_i) - 1]m - [\sum_{i=1}^{m} (k_1 - k_i) - 2]\} + (m-2) \\ &= \sum_{i=1}^{m} n_i + p + (m-2), \end{aligned}$$

where p is an admissible exponent. Hence, by associativity, commutativity, and regularity of the operation $(\cdots)$, there exist $x_1, x_2, \cdots, x_{m-1} \in A$ such that

$$\begin{aligned} 0 \neq a^{n_1} &= (a^{n_1} x_1 a^{n_1} \cdots (\cdots x_{m-1} a^{n_1})) = (a^{mn_1} x_1 \cdots x_{m-1}) \\ &= (a^{n_1 + n_2 + \cdots + n_m + p + (m-2)} x_1 \cdots x_{m-1}) = (a^{n_1 + n_2 + \cdots + n_m} (a^p a a \cdots a x_1) x_2 \cdots x_{m-1}) \\ &= (0 (a^p a a \cdots a x_1) x_2 \cdots x_{m-1}) = 0 \end{aligned}$$

a contradiction. Thus every element of $[a]$ is non-zero and the conclusion follows.

References

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