76. *Note on the Modular Forms*

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1. In his paper [2] Bochner treated the modular forms of level 1. We shall add a little to his result. In the following we shall use freely the notaions and the results in the papers of Bochner and of ourselves [5].

2. By the theory of Bochner we have

Theorem of Bochner. *Let λ and k be positive numbers and $f(z)$ be an analytic function defined on the upper half plane such that $f(z+\lambda)=f(z)$ and $f(z)=\pm\left(\frac{i}{z}\right)^k f\left(-\frac{1}{z}\right)$. Let $\sum_{n=0}^{\infty} a_n e^{\frac{2\pi}{\lambda}nzi}$ be the Fourier series of $f(z)$ and $\sum_{n=0} a_n n^{-s}$ be convergent for some s. Then*

$$\sum_{n=0}^{\infty} a_n \varphi(\sqrt{n}) = \pm \sum_{n=0}^{\infty} a_n T_{\lambda,2k}\varphi(\sqrt{n})$$

for any φ in $\mathfrak{P}_0$, where $T_{\lambda,2k}\varphi$ is the Bochner transform of φ.

From now we shall consider the case where $\lambda=1$, k is an even number, $a_0=0$ and $f(z)=z^{-k}f\left(-\frac{1}{z}\right)$. In this case $\sum_{n=1}^{\infty} a_n e^{2\pi nzi}$ is a cusp form of dimension $-k$ and of level 1. By the general theory of cusp form (Hecke [4] p. 652) we know $\sum_{n=1} \frac{a_n}{n^s}$ converges absolutely for $\operatorname{Re} s > \frac{k+1}{2}$. Using the above theorem of Bochner we can prove

Proposition 1. *Let k be an even natural number and $\sum_{n=1}^{\infty} a_n e^{2\pi nxi}$ be a cusp form of dimension $-k$ and of level 1. If $f(x)$ is a function of class C^∞ such that $\sum_{n=1}^{\infty} a_n f(\sqrt{n})$ is convergent and $\int_0^\infty x^{k+\frac{3}{2}} \left| \left(\frac{d}{xdx}\right)^2 f(x) \right| dx$ exists, then*

$$\sum_{n=1}^{\infty} a_n f(\sqrt{n}) = (-1)^{\frac{k}{2}} \sum_{n=1}^{\infty} a_n T_{1,2k} f(\sqrt{n}).$$

Proof. We have $|T_{1,2k}f(\sqrt{n})| = O(n^{-\frac{k}{2}-\frac{3}{4}})$ by Proposition 4 in [5]. Therefore $\sum_{n=1}^{\infty} a_n Tf(\sqrt{n})$ is absolutely convergent by Hecke's theorem.

Now we take functions $\varphi_1(x), \varphi_2(x), \cdots$ in $\mathfrak{P}_0$ such that

$$\begin{aligned} &\varphi_m(x)=f(x) && \text{for } 0\le x\le\sqrt{m}, \\ &\varphi_m(x)=0 && \text{for } x\ge\sqrt{m+1} \text{ and} \\ &|\varphi_m(x)|\le|f(\sqrt{m})| && \text{for } \sqrt{m}<x<\sqrt{m+1}. \end{aligned}$$

Then

$$\sum_{n=1}^{\infty} a_n \varphi_m(\sqrt{n}) = (-1)^{\frac{k}{2}} \sum_{n=1}^{\infty} a_n T_{1,2k} \varphi_m(\sqrt{n}) \text{ and}$$

$$\sum_{n=1}^{\infty} a_n \varphi_m(\sqrt{n}) = \sum_{n=1}^{m} a_n f(\sqrt{n}) \text{ converges to } \sum_{n=1}^{\infty} a_n f(\sqrt{n}).$$

On the other hand we obtain by Proposition 4 in [5]

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n T_{1,2k} f(\sqrt{n}) - \sum_{n=1}^{\infty} a_n T_{1,2k} \varphi_m(\sqrt{n}) \\
&\quad = O\Big(\sum_{n=1}^{\infty} |a_n| n^{-\frac{k}{2}-\frac{3}{4}} \int_0^{\infty} x^{k+\frac{3}{2}} \Big| \Big(\frac{d}{xdx}\Big)^2 (f(x) - \varphi_m(x)) \Big| dx\Big) \\
&\quad = O\Big(\int_{\sqrt{m}}^{\infty} x^{k+\frac{3}{2}} \Big| \Big(\frac{d}{xdx}\Big)^2 f(x) \Big| dx\Big).
\end{aligned}$$

And the last term converges to 0 if m tends to infinity. (Q.E.D.)

3. We shall apply Proposition 1 to the function $f(x) = x^{-\nu} J_\nu(\alpha x)$ with suitable number ν, where α is a positive number. Because $f(\sqrt{n}) = O(n^{-\frac{\nu}{2}} n^{-\frac{1}{4}})$ and $\sum \frac{a_n}{n^s}$ is absolutely convergent for $\operatorname{Re} s > \frac{k+1}{2}$, $\sum_{n=1}^{\infty} a_n f(\sqrt{n})$ converges absolutely if $\nu > k + \frac{1}{2}$. On the other hand, we have

$$\int_0^{\infty} x^{k+\frac{3}{2}} \Big| \Big(\frac{d}{xdx}\Big)^2 f(x) \Big| dx \le c + c \int_1^{\infty} x^{k+\frac{3}{2}-(\nu+2)-\frac{1}{2}} dx,$$

since

$$\Big(\frac{d}{xdx}\Big)^2 f(x) = x^{-(\nu+2)} J_{\nu+2}(\alpha x).$$

Therefore the integral converges for $\nu > k$.

According to Bateman [1] p. 48 (7) we have

$$T_{1,2k}\{x^{-\nu} J_\nu(4\pi\sqrt{\xi}\, x)\} = \begin{cases} \xi^{-\frac{\nu}{2}} \pi^{\nu-k} \dfrac{1}{\Gamma(\nu-k+1)} (\xi - x^2)^{\nu-k} & \text{for } 0 < x < \sqrt{\xi}, \\ 0 & \text{for } x \ge \sqrt{\xi}, \end{cases}$$

if $\nu + 1 > k > 0$ and $\xi > 0$.

Thus we can apply Proposition 1 to $x^{-(k+r)} J_{k+r}(4\pi\sqrt{\xi}\, x)$, where $r > \frac{1}{2}$, and we get

$$\sum_{n=1}^{\infty} a_n n^{-\frac{k+r}{2}} J_{k+r}(4\pi\sqrt{\xi n}) = (-1)^{\frac{k}{2}} \frac{(2\pi)^r \xi^{-\frac{k+r}{2}}}{\Gamma(r+1)} \sum_{0<n<\xi} a_n (\xi - n)^r.$$

Therefore

$$\sum_{0<n<\xi} a_n (\xi - n)^r = O(\xi^{\frac{k+r}{2}-\frac{1}{4}}).$$

Proposition 2. *If* $\sum_{n=1}^{\infty} a_n e^{2\pi n z i}$ *is a cusp form of even dimension* $-k$ *and of level* 1 *and* $r > \frac{1}{2}$, *then*

$$\sum_{n=1}^{\infty} a_n \Big(\frac{\xi}{n}\Big)^{\frac{k+r}{2}} J_{k+r}(4\pi\sqrt{\xi n})$$

is locally uniformly convergent and its value is equal to

$$(-1)^{\frac{k}{2}}\frac{(2\pi)^r}{\Gamma(r+1)}\sum_{0<n<\xi}a_n(\xi-n)^r.$$

(This equality is proved by Bochner in more generalized form. See [2], p. 355, Theorem 11. But only the Abel summability of the infinite series is shown there.)

Corollary. *With the same notation as in Proposition* 2

$$\sum_{0<n<\xi}a_n(\xi-n)^r=O(\xi^{\frac{k}{2}+\frac{r}{2}-\frac{1}{4}})$$

for any real number r greater than $\frac{1}{2}$.

4. We shall now deal with the case $r=0$.

Proposition 3. *The series* $\sum_{n=1}^{\infty}\left(\frac{\xi}{n}\right)^{\frac{k}{2}}a_nJ_k(4\pi\sqrt{n\xi})$ *is uniformly convergent to* $(-1)^{\frac{k}{2}}\sum_{n\leq\xi}a_n$ *in any interval* $[\xi_1,\xi_2]$ *which contains no integer.*

Proof. The method of the proof of this proposition is quite similar to Hardy's in [3]. We begin with stating the results on cusp forms proved by Hecke in [4] (p. 651):

α) $a_n=O(n^{\frac{k}{2}})$,

β) $|a_1|+\cdots+|a_n|=O(n^{\frac{k+1}{2}})$.

and

γ) $\sum_{n=1}^{\infty}a_nn^{-s}$ is absolutely convergent for $s>\frac{k+1}{2}$.

Let us denote

$$A_r(x)=\frac{(-1)^{\frac{k}{2}}}{\Gamma(r+1)}\sum_{n\leq\xi}a_n(x-n)^r$$

for any non-negative number r and

$$A(x)=A_0(x)-(-1)^{\frac{k}{2}}\frac{1}{2}a(x),$$

where $a(x)$ equals to a_x if x is a natural integer and equals to 0 otherwise. Clearly

$$\frac{dA_{r+1}(x)}{dx}=A_r(x)\quad\text{and}\quad\int_0^x A(x)dx=A_1(x).$$

Put

$$S(x,N)=\sum_{n=1}^{N}a_n\left(\frac{x}{n}\right)^{\frac{k}{2}}J_k(4\pi\sqrt{nx}).$$

Then

$$S(x,N)-\left(\frac{x}{N}\right)^{\frac{k}{2}}A_0(N)J_k(4\pi\sqrt{Nx})$$

$$=-x^{\frac{k}{2}}\sum_{n=1}^{N}a_n\int_n^N d(t^{-\frac{k}{2}}J_k(4\pi\sqrt{xt})dt$$

$$=2\pi x^{\frac{k+1}{2}}\sum_{n=1}^{N}a_n\int_n^N t^{-\frac{k+1}{2}}J_{k+1}(4\pi\sqrt{xt})dt$$

$$=2\pi x^{\frac{k+1}{2}}\int_0^N A_0(t)t^{-\frac{k+1}{2}}J_{k+1}(4\pi\sqrt{xt})dt.$$

On the other hand we have

$$\int_0^N A_1(t)t^{-\frac{k+2}{2}}J_{k+2}(4\pi\sqrt{xt})dt$$

$$=-\frac{1}{2\pi\sqrt{x}}A_1(N)N^{-\frac{k+1}{2}}J_{k+1}(4\pi\sqrt{Nx})+\frac{1}{2\pi\sqrt{x}}\int_0^N A_0(t)t^{-\frac{k+1}{2}}J_{k+1}(4\pi\sqrt{xt})dt.$$

Therefore we get

$$S(x,N)=\left(\frac{x}{N}\right)^{\frac{k}{2}}A_0(N)J_k(4\pi\sqrt{Nx})+2\pi\left(\frac{x}{N}\right)^{\frac{k+1}{2}}A_1(N)J_{k+1}(4\pi\sqrt{Nx})$$
$$+4\pi^2x^{\frac{k+2}{2}}\int_0^N A_1(t)t^{-\frac{k+2}{2}}J_{k+2}(4\pi\sqrt{xt})dt.$$

By the estimation (β) and $J_\nu(z)=O(z^{-\frac{1}{2}})$ the first term on the right hand side is equal to $O(N^{\frac{1}{4}}\log N)$ locally uniformly for x. And by Corollary of Proposition 2 the second term is equal to $O(N^{-\frac{1}{2}})$ locally uniformly for x.

Let us denote the last term with $K(x,N)$. Then we have

$$S(x,N)=K(x,N)+o(1)$$

and

$$K(x,N)=(-1)^{\frac{k}{2}}2\pi x^{\frac{k+2}{2}}\sum_{n=1}^{\infty}a_n n^{-\frac{k+1}{2}}\int_0^N t^{-\frac{1}{2}}J_{k+2}(4\pi\sqrt{xt})J_{k+1}(4\pi\sqrt{nt})dt$$

$$=(-1)^{\frac{k}{2}}x^{\frac{k+1}{2}}\sum_{n=1}^{\infty}a_n n^{-\frac{k+1}{2}}\int_0^{4\pi\sqrt{xN}}J_{k+2}(u)J_{k+1}\left(\sqrt{\frac{n}{x}}u\right)du.$$

Because $$\int_0^\infty J_{\nu+1}(x)J_\nu(ax)=\begin{cases}a^\nu & (0<a<1)\\ \dfrac{1}{2} & (a=1)\\ 0 & (a>1),\end{cases}$$

we have

$$K(x,N)=A(x)-(-1)^{\frac{k}{2}}x^{\frac{k+1}{2}}\sum_{n=1}^{\infty}a_n n^{-\frac{k+1}{2}}\int_{4\pi\sqrt{xN}}^{\infty}J_{k+2}(u)J_{k+1}\left(\sqrt{\frac{n}{x}}u\right)du.$$

Now we shall show that if N tends to the infinity $K(x,N)=A(x)+o(1)$ uniformly in any interval $[x_1,x_2]$ which contains no integer. Since

$$J_\nu(z)=\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+O(z^{-\frac{3}{2}}),$$

we have

$$J_{k+2}(u)J_{k+1}\left(\sqrt{\frac{n}{x}}u\right)=\frac{2}{\pi}x^{\frac{1}{4}}n^{-\frac{1}{4}}u^{-1}\cos\left(u-\frac{k}{2}\pi-\frac{\pi}{4}\right)$$
$$\times\cos\left(\sqrt{\frac{n}{x}}u-\frac{k}{2}\pi-\frac{3}{4}\pi\right)+O(n^{-\frac{1}{4}}u^{-2})$$

uniformly for x in $[x_1, x_2]$. Therefore

$$
\begin{aligned}
K(x, N)-A(x) \\
&=(-1)^{\frac{k}{2}}\frac{1}{\pi}x^{\frac{k}{2}+\frac{3}{4}}\sum_{n=1}^{\infty}a_n n^{-\frac{k}{2}-\frac{3}{4}}\int_{4\pi\sqrt{Nx}}^{\infty}u^{-1}\Big((-1)^{k+1}\cos\Big(\Big(\sqrt{\frac{n}{x}}+1\Big)u\Big) \\
&\quad -\sin\Big(\Big(\sqrt{\frac{n}{x}}-1\Big)u\Big)\Big)du+O\Big(\sum_{n=1}^{\infty}|a_n|n^{-\frac{k+1}{2}-\frac{1}{4}}\cdot\frac{1}{4\pi\sqrt{Nx}}\Big) \\
&=(-1)^{\frac{k}{2}}\frac{1}{\pi}x^{\frac{k}{2}+\frac{3}{4}}\sum_{n=1}^{\infty}a_n n^{-\frac{k}{2}-\frac{3}{4}}\Big((-1)^{k-1}\int_{4\pi\sqrt{N}(\sqrt{x}+\sqrt{n})}^{\infty}\frac{\cos u}{u}du \\
&\quad -\mathrm{sgn.}(\sqrt{n}-\sqrt{x})\int_{4\pi\sqrt{N}|\sqrt{n}-\sqrt{x}|}^{\infty}\frac{\sin u}{u}du\Big)+O(N^{-\frac{1}{2}}),
\end{aligned}
$$

where sgn (0) means 0, and we obtain

$$
\begin{aligned}
K(x, N)-A(x)&=O\Big(\sum_{n=1}^{\infty}|a_n|n^{-\frac{k}{2}-\frac{3}{4}}(N^{-\frac{1}{2}}(\sqrt{x}+\sqrt{n})^{-1}+N^{-\frac{1}{2}}|\sqrt{n}-\sqrt{x}|^{\frac{1}{2}}\Big) \\
&\quad +O(N^{-\frac{1}{2}})=O(N^{-\frac{1}{2}})
\end{aligned}
$$

uniformly for x in $[x_1, x_2]$ if this interval contains no integer. Thus we have proved Proposition 3.

References

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