8. On Newman Algebras. II

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3. The Equational Basis B. To show the equational completeness of system **B**, it will suffice to derive $\bar{N}_2^{+}$ from it, because the ${}^{+\bullet}$-transforms of the equations of **B** and $\bar{N}_2^{+}$ yield precisely Wooyenaka's axiom system II (see [7] and [8]):

$$\begin{array}{ccc} & N_1 & \\ & | & \\ & N_1^{\bullet} & \\ / & & \backslash \\ N_2^{+\bullet} & & \bar{N}_2^{+\bullet} \\ \backslash & & / \\ & N_2^{+} & \\ & | & \\ & \bar{N}_2^{\bullet}. & \end{array}$$

This implies then that $\mathbf{B}^{+\bullet}$ is an equational basis for Newman algebras and the superfluousness of $\bar{N}_2^{\bullet}$ in Wooyenaka's system II.

3.1. $xx=x$.

$x=x(x+\bar{x})=xx+x\bar{x}=xx$ (N_2, N_1, $\bar{N}_2$).

3.2. $x\bar{\bar{x}}=\bar{\bar{x}}$.

$x\bar{\bar{x}}=x\bar{\bar{x}}+\bar{x}\bar{\bar{x}}=(x+\bar{x})\bar{\bar{x}}=\bar{\bar{x}}$ ($\bar{N}_2$, $N_1^{\bullet}$, $N_2^{\bullet}$).

3.3. $x+\bar{x}=y+\bar{y}$.

$x+\bar{x}=(x+\bar{x})(y+\bar{y})=y+\bar{y}$ (N_2, $N_2^{\bullet}$).

3.4. $x+\bar{x}=\bar{x}+x$.

(a) $(\bar{x}+x)\bar{\bar{x}}=\bar{x}\bar{\bar{x}}+x\bar{\bar{x}}=\bar{x}\bar{\bar{x}}+\bar{\bar{x}}=\bar{x}\bar{\bar{x}}+\bar{\bar{x}}\bar{\bar{x}}=(\bar{x}+\bar{\bar{x}})\bar{\bar{x}}=\bar{\bar{x}}$ ($N_1^{\bullet}$, 3.2, 3.1, $N_1^{\bullet}$, $N_2^{\bullet}$).

(b) $(\bar{x}+x)\bar{x}=\bar{x}\bar{x}+x\bar{x}=\bar{x}\bar{x}=\bar{x}$ ($N_1^{\bullet}$, $\bar{N}_2$, 3.1).

Then $x+\bar{x}=\bar{x}+\bar{\bar{x}}=(\bar{x}+x)\bar{x}+(\bar{x}+x)\bar{\bar{x}}=(\bar{x}+x)(\bar{x}+\bar{\bar{x}})=\bar{x}+x$ (3.3, (a)–(b), N_1, N_2).

3.5. $\bar{\bar{x}}=x$.

$\bar{\bar{x}}=x\bar{\bar{x}}=x\bar{\bar{x}}+x\bar{x}=x(\bar{\bar{x}}+\bar{x})=x(\bar{x}+\bar{\bar{x}})=x$ (3.2, $\bar{N}_2$, N_1, 3.4, N_2).

3.6. $(y\bar{y})\overline{(y\bar{y})}=y\bar{y}$.

$$(y\bar{y})\overline{(y\bar{y})}=(y\bar{y})\overline{(y\bar{y})}+y\bar{y}=(y\bar{y})\overline{(y\bar{y})}+(y\bar{y})^2=(y\bar{y})(\overline{y\bar{y}}+y\bar{y})$$
$$=(y\bar{y})(y\bar{y}+\overline{y\bar{y}})=y\bar{y}\ (\bar{N}_2,\ 3.1,,\ N_1,\ 3.4,\ N_2).$$

3.7. $\overline{y\bar{y}}=y+\bar{y}$.

$\overline{y\bar{y}}=(y\bar{y}+\overline{y\bar{y}})\overline{(y\bar{y})}=(y\bar{y})\overline{(y\bar{y})}+\overline{(y\bar{y})}^2=y\bar{y}+\overline{y\bar{y}}=y+\bar{y}$ ($N_2^{\bullet}$, $N_1^{\bullet}$, 3.6–3.1, 3.3).

3.8. $x\bar{x}=y\bar{y}$ (3.5, 3.7, 3.3, 3.7, 3.5).

3.9. $x(y\bar{y})=y\bar{y}$.

$$x(y\bar{y})=x(x\bar{x})=x(x\bar{x})+x\bar{x}=x(x\bar{x}+\bar{x})=x(x\bar{x}+\bar{x}\bar{x})$$
$$=x((x+\bar{x})\bar{x})=x\bar{x}=y\bar{y}\ (3.8,\ \bar{N}_2,\ N_1,\ 3.1,\ N_1^{\bullet},\ N_2^{\bullet},\ 3.8)$$

3.10. $y\bar{y}+x=x$.

$y\bar{y}+x=x(y\bar{y})+x(y+\bar{y})=x(y\bar{y}+(y+\bar{y}))=x(y\bar{y}+\overline{\overline{y\bar{y}}})=x$ (3.9–N_2, N_1, 3.7, N_2).

BN_1. The indpendence of N_1 in **B** is shown by the model $\bar{P}$ of Y. Wooyenaka [8] page 86.

B$N_1^{\cdot}$. The independence-model of $N_1^{\cdot}$ in **B** is obtained from the preceding model by Wooyenaka by transposing its $+$-table and $\cdot$-table.

BN_2. The following model proves the independence of N_2 from the rest of **B**:

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	1
1	0	1

y	$\bar{y}$
0	1
1	0

Observe here that $1(0+\bar{0})\neq 1$.

B$N_2^{\cdot}$. The model for independence of $N_2^{\cdot}$ in B is obtained from **B**N_2 by transposing its $\cdot$-table.

B$\bar{N}_2$. The following is a model for $\bar{N}_2$'s independence from the rest of **B**:

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

y	$\bar{y}$
0	1
1	1

Here note that $0+1\bar{1}\neq 0$.

4. The Equational Basis C. To show the adequacy of **C** as a formulation of Newman algebras, we shall derive $\bar{N}_2^{+}$ and N_6 (and hence **A**) from it.

4.1. $x+y\bar{y}=x$.

$x=x(x+\bar{x})=xx+x\bar{x}=x+y\bar{y}$ (N_2, N_1, N_5-N_8).

4.2. $x+\bar{x}=y+\bar{y}$ (N_2, N_3, N_2).

4.3. $\bar{\bar{x}}x=\bar{\bar{x}}$.

$\bar{\bar{x}}=\bar{\bar{x}}(x+\bar{x})=\bar{\bar{x}}x+\bar{\bar{x}}\bar{x}=\bar{\bar{x}}x+\bar{x}\bar{\bar{x}}=\bar{\bar{x}}x$ (N_2, N_1, N_3, 4.1).

4.4. $\bar{x}+x=x+\bar{x}$.

From the identities (a) $\bar{\bar{x}}=\bar{\bar{x}}(\bar{x}+\bar{\bar{x}})=\bar{\bar{x}}\bar{x}+\bar{\bar{x}}^2=\bar{\bar{x}}\bar{x}+\bar{\bar{x}}=\bar{\bar{x}}\bar{x}+\bar{\bar{x}}x=\bar{\bar{x}}(\bar{x}+x)=(\bar{x}+x)\bar{\bar{x}}$ (N_2, N_1, N_5, 4.3, N_1, N_3) and (b) $\bar{x}=\bar{x}+x\bar{x}=\bar{x}\bar{x}+\bar{x}x=\bar{x}(\bar{x}+x)=(\bar{x}+x)\bar{x}$(4.1, N_5-N_3, N_1, N_3), we obtain $x+\bar{x}=\bar{x}+\bar{\bar{x}}=(\bar{x}+x)\bar{x}+(\bar{x}+x)\bar{\bar{x}}=(\bar{x}+x)(\bar{x}+\bar{\bar{x}})=\bar{x}+x$(4.2, (a)-(b), N_1, N_2).

4.5. $\bar{\bar{x}}=x$.

$\bar{\bar{x}}=\bar{\bar{x}}x=\bar{\bar{x}}x+x\bar{x}=x\bar{\bar{x}}+x\bar{x}=x(\bar{\bar{x}}+\bar{x})=x(\bar{x}+\bar{\bar{x}})=x$ (4.3, 4.1, N_8, N_1, 4.4, N_2).

4.6. $y\bar{y}+x=x$.

$$y\bar{y}+x=x\bar{x}+xx=x(\bar{x}+x)=x(x+\bar{x})=x \quad (N_8-N_5,\ N_1,\ 4.4,\ N_2).$$

$\mathbf{C}N_1$. *Independence-Model of N_1 in* **C**.

$+$	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	1	1
b	b	1	1	1

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	0
b	0	b	0	b

y	$\bar{y}$
0	1
1	0
a	b
b	a

Note, $a(b+b)\neq ab+ab$.

$\mathbf{C}N_2$ is the same as $\mathbf{A}N_2$.

$\mathbf{C}N_3$ is the same as $\mathbf{A}N_3$.

$\mathbf{C}N_5$. *Independence-Model of N_5 in* **C**.

$+$	0	1	a
0	a	1	0
1	1	0	a
a	0	a	1

$\cdot$	0	1	a
0	a	0	1
1	0	1	a
a	1	a	0

y	$\bar{y}$
0	1
1	0
a	a

Here we have $aa\neq a$, $00\neq 0$.

$\mathbf{C}N_8$. *Independence-Model of N_8* in **C**.

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

y	$\bar{y}$
0	1
1	1

Observe that $0\bar{0}\neq 1\bar{1}$.

5. **The Equational Basis D.** This time, we shall derive N_2 and N_8 (and hence **C**) from **D**.

5.1. $x(y+\bar{y})=x$.

$$x(y+\bar{y})=x(x+\bar{x})=xx+x\bar{x}=xx=x \quad (\bar{N}_8,\ N_1,\ \bar{N}_2,\ N_5).$$

The following propositions are derived in exactly the same way as in section 4 (propositions 4.3, 4.4, 4.5):

5.2. $\bar{\bar{x}}x=\bar{\bar{x}}$.

5.3. $\bar{x}+x=x+\bar{x}$.

5.4. $\bar{\bar{x}}=x$.

5.5. $\overline{y\bar{y}}=y+\bar{y}$

$$\overline{y\bar{y}}=\overline{y\bar{y}}+y\bar{y}=y\bar{y}+\overline{y\bar{y}}=y+\bar{y} \quad (\bar{N}_2,\ 5.3,\ \bar{N}_8).$$

5.6. $x\bar{x}=y\bar{y}$ (5.4, 5.5, $\bar{N}_8$, 5.5, 5.4).

$\mathbf{D}N_1$ is the same as that of $\mathbf{C}N_1$.

D$\bar{N}_2$, the independence-model of $\bar{N}_2$ in **D**, is the following:

+	0	1
0	0	1
1	0	1

·	0	1
0	0	1
1	1	1

y	$\bar{y}$
0	1
1	1

Note here that $0+y\bar{y}\neq 0$.

DN_3 is the same as **A**N_3 and **C**N_3.

DN_5. The independence-model of N_5 in **D** is given by

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	0	0

y	$\bar{y}$
0	1
1	0

In this case, $11\neq 1$.

D$\bar{N}_8$ in the following:

+	0	1
0	0	0
1	0	1

·	0	1
0	0	1
1	1	1

y	$\bar{y}$
0	1
1	1

Note, $0+\bar{0}\neq 1+\bar{1}$.

6. **The Equational Basis E.** It is sufficient to derive N_2, and hence **B**, in order to show its equational completeness.

6.1. $x\bar{x}=y\bar{y}$.

$x\bar{x}=x\bar{x}+y\bar{y}=y\bar{y}$ ($\bar{N}_2$, $\bar{N}_2^+$).

6.2. $\overline{x+\bar{x}}=x\bar{x}$.

$\overline{x+\bar{x}}=(x+\bar{x})\overline{(x+\bar{x})}=x\bar{x}$ ($N_2^{\cdot}$, 6.1).

6.3. $x+\bar{x}=y+\bar{y}$.

$x+\bar{x}=\overline{\overline{x+\bar{x}}}=\overline{x\bar{x}}=\overline{y\bar{y}}=\overline{\overline{y+\bar{y}}}=y+\bar{y}$ (N_6, 6.2, 6.1, 6.2, N_6).

6.4. $xx=x$.

$x=\bar{\bar{x}}=(x+\bar{x})\bar{\bar{x}}=x\bar{\bar{x}}+\bar{x}\bar{\bar{x}}=x\bar{\bar{x}}=xx$ (N_6, $N_2^{\cdot}$, $N_1^{\cdot}$, $\bar{N}_2$, N_6).

6.5. $x(y+\bar{y})=x$.

$x(y+\bar{y})=x(x+\bar{x})=xx+x\bar{x}=xx=x$ (6.3, N_1, $\bar{N}_2$, 6.4).

EN_1 and **E**$N_1^{\cdot}$ are respectively the same models **B**N_1 and **B**$N_1^{\cdot}$ (by Y. Wooyenaka).

E$\bar{N}_2$ is the same as **A**N_2.

E$\bar{N}_2$ is the following:

+	0	1
0	0	1
1	0	1

·	0	1
0	0	1
1	0	1

y	$\bar{y}$
0	1
1	0

In this case, $0+0\bar{0}\neq 0$.

$\mathbf{E}\bar{N}_2^+$ is the same as $\mathbf{B}N_2$. In this case, note that $0\bar{0}+0\neq 0$.

$\mathbf{E}N_6$ is given by the following:

+	0	1
0	0	0
1	0	1

$\cdot$	0	1
0	0	1
1	0	1

y	$\bar{y}$
0	1
1	0

Here we have $\bar{\bar{0}}\neq 0$.

7. Concluding Remarks. As we have previously observed [6], every postulate-system for Newman algebras gives rise to a postulate-system for Boolean algebras when any one of the following equations is added as an additional postulate: $x+x=x$, $x+yz=(x+y)(x+z)$, $x+xy=x$, $x(x+y)=x$, $x+(y+\bar{y})=y+\bar{y}$, $(\bar{y}+y)+y=\bar{y}$. In the cases of **A**, **B**, **D** or **E** together with $x+x=x$, it is easy to see that we obtain, in fact, equational bases for Boolean algebras. Similarly, if the equation $(xx)y=x(xy)$ were added to any postulate system for Newman algebras, then a postulate-system for Boolean rings with identity is obtained.

References

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