61. *On Linear Isotropy Group of a Riemannian Manifold*

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Introduction. Let M be a connected Riemannian manifold of dimension n and of class C^∞, and let M_p be the tangent space of M at p. According to the Riemannian structure a scalar product $g_p(X, Y)$ is defined for any vectors $X, Y \in M_p$. We denote by L_p the group of all linear transformations of M_p. The *infinitesimal linear isotropy group* K_p is, by definition [2], the subgroup of L_p consisting of all linear transformations of M_p which leave invariant the curvature tensor R and the successive covariant differentials ∇R, $\nabla^2 R, \cdots$ at p. We define a group A_p as a subgroup of K_p consisting of all elements of K_p which leave invariant the scalar product $g_p(X, Y)$. Let $I(M)$ be the group of isometries of M. Let H_p be the isotropy group of $I(M)$ at p, and let dH_p be the linear isotropy group of H_p. In §1, we shall investigate sufficient conditions that $dH_p = A_p$. §2 is devoted to applications of the main theorem to Riemannian globally symmetric spaces.

§1. Main theorem.

Theorem 1. *If M is a simply connected homogeneous Riemannian manifold, then $dH_p = A_p$ for each p in M.*

In order to prove this theorem, we need the following:

Lemma. *If M is an analytic complete simply connected Riemannian manifold, then $dH_p = A_p$ for each p in M.*

Proof. We have proved that $dH_p \subset A_p$ for any Riemannian manifold [3] p. 1). Take a normal coordinate system $\{x_1, \cdots, x_n\}$ at p, with coordinate neighborhood U. We may assume that $\{(\partial/\partial x_1)_p, \cdots, (\partial/\partial x_n)_p\}$ is an orthonormal base, and that U is the interior of a gedesic sphere centered at p. U has the Riemannian metric induced from M. Since M is analytic, each element $a \in A_p$ induces a local isometry $\tilde{f}$ which maps U onto itself, such that $\tilde{f}(p) = p$ and $(d\tilde{f})_p = a$ ([3] p. 2). Since M is a simply connected complete analytic Riemannian manifold, and U is a connected open subset of M, this local isometry $\tilde{f}$ can be uniquely extended to f, an isometry of M ([4] p. 256). Clearly $f(p) = p$ and $(df)_p = a$. Therefore we have $A_p \subset dH_p$.

Proof of Theorem. Since M is a Riemannian homogeneous space of a Lie group, it can be considered to be an analytic complete Riemannian manifold. Since M is simply connected it satisfies the conditions of the lemma.

Counterexample. Consider in E^3 a cylinder of revolution with the natural Riemannian metric from Euclidean metric in M. This is a homogeneous Riemannian manifold, which is not simply connected. In this case, $dH_p=$ identity and A_p is the rotation group of E^2. This example shows that the simply connectedness of the theorem can not be removed.

Corollary. *If M is an analytic complete simply connected Riemannian manifold, then H_p is isomorphic to dH_p as Lie groups.*

Proof. Let U be the neighborhood with the same Riemannian structure as in above lemma. Let $\tilde{H}_p$ be the group of all isometries of U which fix the point p. Then each element $f \in H_p$ induces $f_{|U} \in \tilde{H}_p$. Since M is a simply connected analytic complete Riemannian manifold, each $\tilde{f} \in \tilde{H}_p$ can be extended uniquely to f, an isometry of M. Clearly $f_{|U}=\tilde{f}$. Therefore H_p is isomorphic to $\tilde{H}_p$ as Lie groups. Each element of A_p can be expressed by a matrix with respect to the base $\{(\partial/\partial x_1)_p, \cdots, (\partial/\partial x_n)_p\}$. In this coordinate system $\{x_1, \cdots, x_n\}$ each element of $\tilde{H}_p$ can be expressed by

$$y_i=\sum_{j=1}^{n} a_{ij}x_j (i=1, 2, \cdots, n),$$

where the matrix $\|a_{ij}\|$ belongs to A_p([3] p. 3). This means that $\tilde{H}_p$ is isomorphic to A_p as Lie groups. But $A_p=dH_p$. Therefore H_p is isomorphic to dH_p, and this isomorphism is given by the correspondence $f \in H_p \to (df)_p \in dH_p$.

§ 2. Applications.

In 1927, E. Cartan proved the following theorem ([1] p. 84).

Let M be an affine locally symmetric space without torsion. If a linear transformation of M_p leaves invariant the curvature tensor R at p, then this induces a local affine isomorphism on M.

We shall treat this problem globally imposing some conditions on M.

Theorem 2. *If M is a simply connected Riemannian globally symmetric space, then $dH_p=A_p$.*

Proof. M is a simply connected homogeneous Riemannian manifold. Since M is locally symmetric, the tensors $\nabla^k R$ vanish for $k=1, 2, \cdots,$. By Theorem 1 the conclusion follows.

A Riemannian globally symmetric space M is said to be of the *non-compact type*, if the Riemannian symmetric pair (G, K) is of the noncompact type ([5] p. 194), where G is the identity component of $I(M)$ and K is the isotropy group of G at some point in M. Let us fix a point p and let A the space of A_p.

Theorem 3. *If M is a Riemannian globally symmetric space of the noncompact type, then the space of $I(M)$ is diffeomorphic to $E^n \times A$.*

Proof. For a Riemannian symmetric space of the noncompact type M, there is a normal coordinate system whose coordinate neighborhood is M([5] p. 215). This means that M is diffeomorphic to E^n. Let $0(M)$ be the bundle of orthonormal frames over M, and let F be an orthonormal frame at p. For each $q \in M(q \neq p)$ we put $f_q = \tau_{qp}F$ where τ_{qp} is the parallel translation along the unique geodesic segment from p to q. Therefore we get a C^∞ cross-section in the principal bundle $0(M)$, so that this bundle is equivalent to a product bundle. Each member of $I(M)$ induces a diffeomorphism on $0(M)$ in the natural way. Then the set of frames $I(M)F$ can be considered as a reduced bundle of $0(M)$. Clearly the bundle $I(M)F$ is equivalent to a product bundle. In this bundle, the base space is diffeomorphic to E^n, and the standard fiber is diffeomorphic to A.

Corollary. *If M is a Riemannian globally symmetric space of the non-compact type, then $dH_p = A_p$.*

Proof. Since M is simply connected, by Theorem 2 the conclusion follows.

References

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