150. On Indefinite (E. R.)-Integrals. II

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§3. Now, let us prove the following main theorem.

Theorem. If $f(x)$ is $\mathscr{D}$-integrable in $I_0=[a, b]$, there exists a measure ν such that $f(x)$ has a indefinite (E.R. ν)-integral, (E.R. ν) $\int_a^x f(t)dt$, and (E.R. ν) $\int_a^x f(t)dt=(\mathscr{D})\int_a^x f(t)dt$ for all $x \in I_0$.

Proof. We may clearly assume that $f(x)=0$ for all $x \in C(I_0)$. If the function $f(x)$ is summable on I_0, we have (E.R. ν) $\int_a^x f(t)dt=\int_a^x f(t)dt=(\mathscr{D})\int_a^x f(t)dt$ for every measure ν which fulfils condition 1*) and 2*) [1].

Next, we shall consider the case in which $f(x)$ is not summable. Let $f(x)$ be a function which is $\mathscr{D}$-integrable but not summable on I_0. Then, there exists, by the lemma, a non-decreasing sequence of closed sets $\{F_l\}$ such that (i) $\bigcup_{l=1}^{\infty} F_l=I_0$, (ii) $f(x)$ is summable on F_l,

$$\text{(iii)}\quad |F(I)-\int_{F_l\cap I} f(x)dx| \leq 2^{-l} \text{ for every interval } I\subset I_0, \tag{1}$$

$$\text{(iv)}\quad \sum_{j=1}^{\infty} |F(J_l^j)| \leq 2^{-l} \tag{2}$$

for the sequence of intervals $\{J_l^j\}$ contiguous to the closed set which consists of all points of F_n and end points of I_0.

Since $f(x)$ is by hypothesis, not summable, we may assume that

$$\int_{F_l-F_{l-1}} |f(x)|\, dx \geq 2^{-l} \qquad l=1, 2, 3 \cdots$$

$$\text{(we regard } F_0 \text{ as empty).} \tag{3}$$

On account of this and summability of $f(x)$ on F_l, we find, for every l, a measurable set $H_l\subset F_l$ such that $f(x)\geq f(x')$ for every $x \in H_l$ and $x' \in F_l-H_l$, and

$$\int_{H_l} |f(x)|\, dx=2^{-l}. \tag{4}$$

Writing $\delta_l=\text{mes}\, H_l$, we see at once that

$$\text{mes}\,(F_l-F_{l-1})>\delta_l, \tag{5}$$

$$\delta_l>\delta_{l+1}, \tag{6}$$

$$\text{mes}\,(E)<\delta_l \text{ implies } \int_E |f(x)|\, dx\leq 2^{-l} \tag{7}$$

for every measurable set $E\subset F_l$.

Let h_l and k_l be integers such that

$$(h_l-1)\delta_l< \text{mes}\,(F_l-F_l-1)<h_l\delta_l, \tag{8}$$

$$2^{k_l-1}\delta_{l+1}<\delta_l<2^{k_l}\delta_{l+1}\,. \tag{9}$$

Then, h_l and k_l are uniquely determined and we have, by (5) and (6), $h_l \geq 2$ and $k_l \geq 1$. Hence there exists, for each n, a integer $l(n)$ such that $\sum_{l=1}^{l(n)-1}(h_l+k_l)<n\leq\sum_{l=1}^{l(n)}(h_l+k_l)$. Writing $m(n)=n-\sum_{l=1}^{l(n)-1}(h_l+k_l)$, we have

$$n=\sum_{l=1}^{l(n)-1}(h_l+k_l)+m(n),\ 1\leq m(n)\leq h_{l(n)}+k_{l(n)}. \tag{10}$$

Let $a=a_l^0<\cdots<a_l^m<a_l^{m+1}<\cdots<a_l^{h_l+k_l}=b$ be a sequence such that

$$\operatorname{mes}(F_l-F_{l-1})\cap[a_l^{m-1}, a_l^m]=\begin{cases}\operatorname{mes}(F_l-F_{l-1})/h_l & \text{when } 1\leq m\leq h_l-1\\ \operatorname{mes}(F_l-F_{l-1})/h_l\cdot 2^{(m-h_l+1)} & \text{when } h_l<m\leq h_l+k_l-1\\ \operatorname{mes}(F_l-F_{l-1})/h_l\cdot 2^{-k_l} & \text{when } m=h_l+k_l.\end{cases} \tag{11}$$

Writing $E_0=(-\infty, a)\cup(b, +\infty)$, $E_n=\{F_{l(n)}-F_{l(n)-1}\}\cap(a^{m(n)-1}, a^{l(n)}]$ for $n\geq 1$, we have $E_n\cap E_{n'}=\phi$ for $n\neq n'$ and $\bigcup_{n=1}^{\infty}E_n=\bigcup_{l=1}^{\infty}(F_l-F_{l-1})-\{a\}=I_0-\{a\}$. Hence, if we define a measure ν by the relation $\nu(E)=\sum_{n=0}^{\infty}2^{-n}\operatorname{mes}(E\cap E_n)$ for every measurable set E, $\nu(E)$ is a measure which fulfills the conditions 1*) and 2*).

Now we shall show that the sequence $V(\varepsilon_n, A_n:f_n)$ which defined by the relations that

$$\varepsilon_n=(|I_0|+1)2^{-l(n)}$$
$$A_n=\bigcup_{i=1}^{n}E_i\cup(-\infty, a]\cup(b, \infty)$$
$$f_n=\begin{cases}f(x) & \text{for } x\in A_n\\ 0 & \text{for } x\in C(A_n)\end{cases}$$

is canchy sequence converge to $f(x)$.

It is easily seen that $\varepsilon_n\downarrow 0$ and that A_n is a non-decreasing sequence of closed sets such that $\bigcup_{n=1}^{\infty}A_n=(-\infty, \infty)$. It follows that $V(\varepsilon_n, A_n; f_n)\supset(\varepsilon_{n+1}, A_{n+1}; f_{n+1})$ and $f(x)\in V(\varepsilon_n, A_n: f_n)$ for every n. On account of (11), (8), and (6), we have

$$\nu(A_n^c)=\sum_{i=n+1}^{\infty}\nu(E_i)=\sum_{i=n+1}^{\infty}2^{-i}\operatorname{mes}(E_i)\leq\sum_{i=n+1}^{\infty}2^{-i}\delta_{l(n)}$$
$$=2^{-n}\delta_{l(n)}<\varepsilon_n \qquad \text{for every } n.$$

Since $\nu(B)\geq 2^{-n}\operatorname{mes}(B)$ for every $B\subset A_n$, $\nu(B)\leq\nu(C(A_n))$ implies $\operatorname{mes}(B\cap A_n)\leq 2^n\nu(B)\leq 2^n\cdot 2^{-n}\cdot\delta_{l(n)}=\delta_{l(n)}$. Hence, by (7), we have, for every measurable set B

$$\nu(B)\leq\nu(C(A_n)) \text{ implies } \int_B|f(x)|\,dx$$
$$=\int_{B\cap A_n}|f(x)|\,dx\leq 2^{-l(n)}<\varepsilon_n.$$

It is easily seen that $\nu(E_{n+1})\geq\frac{1}{4}\nu(E_n)$. It follows that $\nu(C(A_{n+1})\geq \nu(E_{n+2})\geq\frac{1}{4}\nu(E_{n+1})$. Hence, we have, for every n,

$$\nu(c(A_n))=\nu(c(A_{n+1}))+\nu(E_{n+1})\leq\nu(c(A_{n+1})) + 4\nu(c(A_{n+1}))=5\nu(c(A_{n+1})).$$

Finally, we shall show that $\int_I f_n(x)dx$ tend to $F(I)$ for every interval $I\subset I_0$. Let $I=[x_1, x_2]$ be a interval contained in I_0 and let $l=l(n)$, $m=m(n)$. Then, writing $I_1=I\cap[a, a_m^n]$, $I_2=I\cap[a_m^n, b]$, we have $A_n\cap I=(F_l\cap I_1)\cup(F_{l-1}\cap I_2)$. Hence we have, by (1),

$$\left|F(I)-\int_I f_n(x)dx\right|\leq\left|F(I_l)-\int_{I_1\cap F_l}f(x)dx\right| + \left|F(I_2)-\int_{I_2\cap F_{l-1}}f(x)dx\right|\leq 2^{-l}+2^{-(l-1)}=3\cdot2^{-l}.$$

Since $\lim_{n\to\infty} l(n)=0$, $\int_I f_n(x)\,dx$ tend to $F(I)$ for every interval $I\subset I_0$. This complete the proof.

Corollary. If $f_n(x)$ is $\mathscr{D}$-integrable on $I_0=[a, b]$, there exists a measure ν and a cauchy sequence $V(\varepsilon_n, A_n; f_n)\in\mathfrak{C}(f;\nu)$ such that (i) A_n is a non-decreasing sequence of closed sets such that $\bigcup_{n=1}^{\infty} A_n=(-\infty,\infty)$, (ii) $\sum_{i=1}^{\infty}\left|\int_{I_{n_1}^j} f_{n_2}(x)dx\right|\leq\varepsilon_{n_1}$ for every n_2 and n_1, where $\{I_{n_1}^i\}$ is the sequence of intervals contiguous to A_{n_1},

(iii) $\left|(\mathscr{D})\int_I f(x)dx-\int_I f_n(x)dx\right|<\varepsilon_n$

for every interval $I\subset I_0$.

Proof. Taking $\varepsilon_n=(28+|I_0|)\cdot2^{-l(n)}$ and taking A_n, f_n, $l(n)$, $m(n)$, F_l etc. as in the previous theorem, we need only prove that $V(\varepsilon_n, A_n: f_n)$ fulfills second condition. On account of (1), we have, for every interval J contained in some J_l^j,

$$|F(J)|\leq2^{-l}. \tag{12}$$

It follows at once that,

$$\left|\int_{F_{l'}\cap J}f(x)dx\right|<2^{-l+1} \tag{13}$$

for every interval J contained in some J_l^j and for every l and l'.

For every $l'\geq l$, being

$$\int_{J_l^j\cap F_{l'}}f(x)dx=F(J_l^j)-\sum_{\{j': J_{l'}^{j'}\subset J_l^j\}}F(J_{l'}^{j'}),$$

we have

$$\sum_{j=1}^{\infty}\left|\int_{J_l^j\cap F_{l'}}f(x)dx\right|\leq2^{-l+1}. \tag{14}$$

Now, let $n_2\geq n_1$, $l_k=l(n_k)$, $m=m(n_k)$, $a=a_{l_k}^{m_k}$ $(k=1, 2)$ and let $c_k[d_k]$ be the nearest point of $A_n\cap[a, a_k][A\cap[a_k, b]]$ to a_k respectively

$(k=1, 2)$. Then (c_1, d_1) and (c_2, d_2) are contained in some $J^{j_1}_{l_1-1}$ and $J^{j_2}_{l_2-1}$ respectively. Hence we have, by (13) and (14),

$$\begin{aligned}\sum_{i=1}^{\infty}\left|\int_{I^i_{n_1}} f_{n_2}(x)dx\right| &\leq \sum_{\{i:I^i_{n_1}\subset[a,c_2]\}}\left|\int_{I^i_{n_1}\subset F_{l_2}} f(x)dx\right| \\ &+ \sum_{\{i:I^i_{n_1}\subset[d_2,b]\}}\left|\int_{I^i_{n_1}\cap F_{l_2-1}} f(x)dx\right| + \left|\int_{[c_2,a_2]\cap F_{l_2}} f(x)dx\right| \\ &+ \left|\int_{[a_2,d_2]\cap F_{l_2-1}} f(x)dx\right| \leq \sum_{\{j:J^j_{l_1}\subset[a,c_1]\}}\left|\int_{J^i_{l_1}\cap F_{l_2}} f(x)dx\right| \\ &+ \sum_{\{j:J^j_{l_1-1}\subset[d_1,b]\}}\left|\int_{J^j_{l_1-1}\cap F_{l_2}} f(x)dx\right| + \left|\int_{[c_1,d_1]\cap F_{l_2}} f(x)dx\right| \\ &+ \sum_{\{j:J^j_{l_1}\subset[a,c_1]\}}\left|\int_{J^j_{l_1}\cap F_{l_2-1}} f(x)dx + \sum_{\{j:J^j_{l_1-1}\subset[c_1,b]\}}\right|\int_{J^j_{l_1-1}\cap F_{l_2-1}} f(x)dx\Bigg| \\ &+ \left|\int_{[c_1,d_1]\cap F_{l_2-1}} f(x)dx\right| + \left|\int_{[c_2,a_2]\cap F_{l_2}} f(x)dx\right| \\ &+ \left|\int_{[a_2,d_2]\cap F_{l_2-1}} f(x)dx\right| \leq 28\cdot 2^{-l_1}\leq \varepsilon_{n_1}\end{aligned}$$

for every $n_2>n_1$. When $n_2\leq n_1$,

$$\sum_{i=1}^{\infty}\left|\int_{I^i_{n_1}} f_{n_2}(x)dx\right| \leq \int_{c(A_{n_1})} |f_{n_2}(x)|\,dx=0.$$

This complete the proof.

Example. We shall consider a function which has A-integral and $\mathscr{D}$-integral on I_0 but $(A)\int_{I_0} f(x)dx\neq(\mathscr{D})\int_{I_0} f(x)dx$. And we shall construct a measure ν such that (E.R. ν) $\int_I f(x)dx=(\mathscr{D})\int_I f(x)dx$ for every interval $I\subset I_0$.

Let

$$f(x)=\begin{cases} 2^{4n-3}/4n-3 & \text{for } x\in[2^{-4n+3}+2^{-2n+2},\ 2^{-4n+5}+2^{-2n+2}) \\ 2^{4n-1}/4n-1 & \text{for } x\in[2^{-4n+1}+2^{-2n+2},\ 2^{-4n+3}+2^{-2n+2}) \\ -2^{2n}/2n & \text{for } x\in[2^{-4n+1}+2^{-2n},\ 2^{-4n+1}+2^{-2n+2}) \\ 0 & \text{for } x\in(-\infty,0]\cup[3,+\infty). \end{cases}$$

Then it is easily seen that

$$(\mathscr{D})\int_0^3 f(x)dx=\frac{9}{2}\log 2$$

$$(A)\int_0^3 f(x)dx=3\log 2.$$

Next we shall consider (E.R. ν) integral. Let $\{F_l\}$ be the non-decreasing sequence of slosed sets such that $F_l=(-\infty,0)\cup[2^{-4l+1}+2^{-2l}, +\infty)$. Then, applying the same method as in the theorem, we have a sequence $0=a_l^{h_l+k_l}<\cdots<a_l^{m+1}<a_l^m<\cdots<a_l^1<2^{-4l+5}+2^{-2l+2}$ for every l, and we have $A_n=(-\infty,0]\cup[\max(2^{-4l}+2^{-2l}, a_l^m), +\infty)$ for every n, $l=l(n)$ and $m=m(n)$. It follows at once that

$$\lim_{n\to\infty}\int_0^3 f_n(x)dx = \lim_{n\to\infty}\int_{A_n} f(x)dx = \lim_{\varepsilon\to 0}\int_\varepsilon^3 f(x)dx = (\mathscr{D})\int_0^3 f(x)dx.$$

Reference

[1] H. Okano: Sur une généralisation de l'intégrale et un théorème général de l'intégration par parties.