144. On Axiom Systems of Propositional Calculi. VI

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1965)

In this note, we shall continue to deduce axiom systems quoted in [2], [3] from the Russell axiom system. Our considerations are needed in formulations of some algebraic models of propositional calculi. These formulations will be given in the future papers.

The Russell axiom system is as follows:

1 *CpCqp*,
2 *CCpqCCqrCpr*,
3 *CCpCqrCqCpr*,
4 *CNNpp*,
5 *CCpNpNp*,
6 *CCpNqCqNp*.

We shall first prove that the above axioms imply axiom systems by Frege, and Lukasiewicz (L_1) by using the rules of substitution and detachment.

3 *r/p* **C*1—7,

7 *CqCpp*.

7 *q/CpCqp* **C*1—8,

8 *Cpp*.

6 *p/Nq* **C*8 *p/Nq*—9,

9 *CqCNNq*.

3 *p/Cpq*, *q/Cqr*, *r/Cpr* **C*2—10,

10 *CCqrCCpqCpr*.

10 *r/NNq* **C*9—11,

11 *CCpqCpNNq*.

2 *p/Cpq*, *q/CpNNq*, *r/CNqNp* **C*11—*C*6 *q/Nq*—12,

12 *CCpqCNqNp*.

2 *p/Cpq*, *q/CCqrCpr*, *r/s* **C*2—13,

13 *CCCCqrCprsCCpqs*.

13 *s/CCCprsCCqrs* **C*2 *p/Cqr*, *q/Cpr*, *r/s*—14,

14 *CCpqCCCprsCCqrs*.

3 *p/Cpq*, *q/CCprs*, *r/CCqrs* **C*14—15,

15 *CCCprsCCpqCCqrs*.

2 *p/NNp*, *q/p*, *r/q* **C*4—16,

16 *CCpqCNNpq*.

2 *p/Cpq*, *q/CNNpq* **C*16—17,

17 *CCCNNpqrCCpqr*.

10 q/NNq, r/q *C4 p/q—18,
18 CCpNNqCpq.
2 p/CNpq, q/CNqNNp, r/CNqp *C12 p/Np—C18 p/Nq, q/p—19,
19 CCNpqCNqp.
15 p/CNNpq, q/CNqNp, s/CCpqr *C17—C19 p/Np—20,
20 CCCNqNprCCpqr.
20 p/Np, q/p, r/NNp *C5 p/Np—21,
21 CCNppNNp.
10 q/NNq, r/q *C4 p/q—22,
22 CCpNNqCpq.
22 p/CNpp, q/p *C21—23,
23 CCNppp.
15 p/Np, r/p, s/p *C23—24,
24 CCNpqCCqpp.
10 q/CNqp, r/CNpq *C19 p/q, p/q—C1, q/Nq—25,
25 CpCNpq.
2 p/CpCqr, q/CqCpr, r/s *C3—26,
26 CCCqCprsCCpCqrs.
26 q/Np, r/q, s/CCCpqpp *C24 q/Cpq—C25—27,
27 CCCpqpp.
15 p/Cpq, q/r, r/p, s/p *C27—28,
28 CCCpqrCCrpp.
2 p/CCpqr, q/CCrpp, r/s *C28—29,
29 CCCCrppsCCCpqrs.
28 r/q—30,
30 CCCpqqCCqpp.
29 s/CCprr *C30 p/r, q/p—31,
31 CCCpqrCCprr.
13 q/Cqr, r/Csr, s/CCsqCpCsr *C13 p/s, s/CpCsr—32,
32 CCpCqrCCsqCpCsr.
32 p/Cpq, q/CCprs, r/CCqrs, s/t *C14—33,
33 CCtCCprsCCpqCtCCqrs.
33 q/s, s/r, t/CCpqr *C31—34,
34 CCpsCCCpqrCCsrr.
3 p/Cps, q/CCpqr, r/CCsrr *C34—35,
35 CCCpqrCCpsCCsrr.
35 q/r, r/CqCpr *C1 p/Cpr—36,
36 CCpsCCsCqCprCqCpr.
3 p/Cps, q/CsCqCpr, r/CqCpr *C36 –37,
37 CCsCqCprCCpsCqCpr.

37 s/Cqr, q/Cpq *C10—38,

38 CCpCqrCCpqCpr.

The axioms 1, 4 of the Russell system and theses 9, 12, 38 are axioms of Frege, further axiom 2, theses 23 and 25 are axioms of (L_1). Next we shall prove that (R)-system implies (H)-system, i.e. Hilbert axioms of propositional calculus.

2 p/CNpq, q/CNqp *C19—39,

39 CCNqprCCNpqr.

39 r/CCpqq *C 24 p/q, q/p—40,

40 CCNpqCCpqq.

3 p/CNpq, q1Cpq, r/q *C40—41,

41 CCpqCCNpqq.

Axioms 1, 3 and theses 10, 25, 41 are axioms of (H)-system. Now we shall deduce the axioms system of Mendelson (see [4]). As shown above, we have CpCqp, CCpCqrCCpqCpr. We shall prove the thesis: CCNpNqCCNpqp. We have proved that (R)-system is equivalent to (L_3)-system (see [4]). In the following proof, we use the thesis a) CCNpNqCpq.

3 p/CNpq, q/Cqp, r/p *C 24—42,

42 CCqpCCNpqp.

2 p/CNpNq, q/Cqp, r/CCNpqp *C a)—C 42—43,

43 CCNpNqCCNpqp.

This completes the proof of the deduction of Mendelson system.

Finally, we shall prove that the (R)-system implies the (L_2)-system.

3 q/Np, r/q *C25—44,

44 CNpCpq.

2 p/Np, q/Cpq *C44—45,

45 CCCpqrCNpr.

2 p/q, q/Cpq *C1 p/q, q/p—46,

46 CCCpqrCqr.

2 p/CNpq, q/CCpqq *C40—47,

47 CCCCpqqrCCNpqr.

47 q/r, r/CCpqCCqrr *C35 q/r, s/q—48,

48 CCNprCCpqCCqrr.

2 p/CNpr, q/CCpqCCqrr, r/s *C48—49,

49 CCCCpqCCqrrsCCNprs.

49 s/CCqrCCpqr *C3 p/Cpq, q/Cqr—50,

50 CCNprCCqrCCpqr.

Therefore, theses 45, 46, and 50 are axioms of (L_2)-system. On the thesis 24, see [1]. In a future paper, we shall deduce (S_1) and (S_2) systems from (R) system.

References

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[4] K. Iséki and S. Tanaka: On axiom systems of propositional calculi. V. Proc. Japan Acad., **41** (1965).