104. On Variants of Axiom Systems of Propositional Calculus. II

By Shôtarô TANAKA

(Comm. by Kinjirô KUNUGI, M.J.A., May 12, 1966)

In this paper, we shall show that any axiom system containing the *BCK*-system of propositional calculus may be changed into a new system which has axioms less than the number of the original axioms. Following certain 'combinatory logicians', we put *B* for *CCqrCCpqCpr*, *C* for *CCpCqrCqCpr*, and *K* for *CpCqp*. This system was given by C. A. Meredith. Further Prof. K. Iséki has given the algebraic formulation of the *BCK*-system (see, [5]). For the notations and two rules of inference, see [3].

The axioms of the *BCK*-system are given by the following:

1′ *CCqrCCpqCpr*,

2′ *CCpCqrCqCpr*,

3′ *CpCqp*.

It is well known that these axioms imply (see, [1]),

4′ *CCpqCCqrCpr*,

5′ *CpCCpqq*.

Theorem 1. *If F and G are the formulas in the BCK-system, then CpCqp and CCGCFvCwv imply F and G, where F and G do not contain v and w.*

Proof. We put

1 *CpCqp*,

2 *CCGCFvCwv*.

2 *v/w*, *w/CpCqp* **C*1 *p/G*, *q/F*—*C*1—3,

3 *G*.

1 *p/CpCqp* **C*1—4,

4 *CqCpCqp*.

2 *v/CGF*, *w/G* **C*4 *p/F*, *q/G*—*C*3—*C*3—5,

5 *F*.

Theorem 2. *If F and G are the formulas in the BCK-system, then CCGCFvCwv is a formula in this system.*

Proof. 5′ *p/F*, *q/v* **CF*—1,

1 *CCFvv*.

1′*p/G*, *q/CFv*, *r/v* **C*1—2,

2 *CCGCFvCGv*.

2′ *p/CGCFv*, *q/G*, *r/v* **C*2—*CG*—3,

3 *CCGCFvv*.

4′ $p/CGCFv$, q/v, r/Cwv $*C3—C3'$ p/v, $q/w—4$,

4 $CCGCFvCwv$.

By theorems 1 and 2, we can make many new axiom systems of propositional calculi.

The classical propositional calculus contains axioms of the BCK-system, hence, for example, we have the following axiom systems (see, [6], [7]).

1) $CpCqp$, $CCCpCqrCCpqCprCCCNpNqCqpvCwv$.

2) $CpCqp$, $CCpCqrCCpqCpr$, $CCpqCNqNp$, $CCCNNpp$ $CCpNNpvCwv$.

Similarly, we have the following axiom systems of implicational calculus (see, [4]).

1) $CpCqp$, $CCCCpqCCqrCprCCCCpqppvCwv$.

For the positive implicational calculus we have the following axiom system.

1) $CpCqp$, $CCCCpCpqCpqCCCpqCCqrCprvCwv$.

Further we can mention the following variants of the BCK-system (see, [2]).

1) $CpCqp$, $CCCCqrCCpqCprCCCpCqrCqCprvCwv$.

2) $CpCqp$, $CCCpCCpqqCCCqrCCpqCprvCwv$.

Theorem 3. *If F, G, and H are the formulas in the L_1-system* (see, [4]), *then $CpCqCrp$ and $CCFCGCHvCwv$ imply F, G, and H, where F, G, and H do not contain v and w.*

Proof. we put

1 $CpCqCrp$.

2 $CCFCGCHvCwv$.

2 v/F, $w/CpCqCrp$ $*C1$ p/F, q/G, $r/H—C1—3$,

3 F.

1 $p/CpCqCrp$, q/s, r/t $*C1—4$,

4 $CsCtCpCqCrp$.

4 s/F $*C3—5$,

5 $CtCpCqCrp$.

2 v/CFG, w/F $*C5$ p/G, q/H, r/F, $t/F—C3—C3—6$,

6 G.

2 $v/CFCGH$, w/F $*C4$ p/H, q/F, r/G, s/F, $t/G—C3—C3—C6—7$,

7 H.

Theorem 4. *If F, G, and H are the formulas in the L_1-system, then $CCFCGCHvCwv$ is the formula in this system.*

Proof. It is well known that the L_1-system implies the following

1″ $CCpqCCqrCpr$,

2″ $CpCqp$,

3″ *CpCCpqq*,
4″ *CCpCqrCqCpr*,
5″ *CCqrCCpqCpr*.

We put

1 *F*,
2 *G*,
3 *H*,

where *F*, *G*, and *H* do not contain *v* and *w*.

3″ *p/H*, *q/v* **C*1—4,
4 *CCHvv*.
5″ *q/CHv*, *r/v*, *p/G* **C*4—5,
5 *CCGCHvCGv*.
4″ *p/CGCHv*, *q/G*, *r/v* **C*5—*C*2—6,
6 *CCGCHvv*.
5″ *q/CGCHv*, *r/v*, *p/F* **C*6—7,
7 *CCFCGCHvCHv*.
4″ *p/CFCGCHv*, *q/F*, *r/v* **C*7—*C*1—8,
8 *CCFCGCHvv*.
1″ *p/CFCGCHv*, *q/v*, *r/Cwv* **C*8—*C*2″ *p/v*, *q/w*—9,
9 *CCFCGCHvCwv*.

By the theorems 3 and 4, we have new axiom systems of propositional calculi.

From the above proof we have the following two theorems.

Theorem 5. *If F and G are the formulas in the L_1-system, then CpCqCrp and CCFCGvCwv imply F and G.*

Theorem 6. *If F and G are the formulas in the L_1-system, then CCFCGvCwv is a formula in this system.*

From the above theorems 5 and 6, we have the following axiom systems of the classical two valued propositional calculus.

1) *CpCqCrp*, *CCCCpqCNqCprCCCNpqCCpqqvCwv*.
2) *CpCqCrp*, *CCCNpCpqCCCNprCCqrCCpqrvCwv*.

Lemma. *If CpCqp, F, and G are an axiom system in the BCK-system, then they are changed into a single axiom CCCpCqpCCCFCGvCwvvuCxu which is equivalent to the original one.*

Proof. Let

1 *CCCpCqpCCCFCGvCwvvuCxu*.
1 *p/CpCqp*, *q/CCFCGvCwv*, *u/CpCqp* **C*1 *u/CpCqp*, *x/CCFCGvCwv*—2,
2 *CxCpCqp*.
2 *x/CxCpCqp* **C*2—3,
3 *CpCqp*.
1 *u/CCpCqpCCFCGvCwv*, *x/CpCqp* **C*2 *x/CpCqp*,

$p/CCFCGvCwv$, $q/CpCqp$—$C3$—$C3$—4,

4 $CCFCGvCwv$.

By the above theorem and Theorem 1 we have

5 F.

6 G.

Next we shall prove the converse. Let

1 F,

2 G,

where u, v, w, and x are not contained in F and G.

3′ p/G, q/v *$C1$—3,

3 $CCGvv$.

5′ q/CGv, r/v, p/F *$C3$—4,

4 $CCFCGvCFv$.

4′ $p/CFCGv$, q/F, r/v *$C4$—$C1$—5,

5 $CCFCGvv$.

1′ $p/CFCGv$, q/v, r/Cwv *$C5$—$C2'$ p/v, q/w—6,

6 $CCFCGvCwv$.

3′ $p/CCFCGvCwv$, q/u *$C6$—7,

7 $CCCCFCGvCwvuu$.

5′ $q/CCCFCGvCwvu$, r/u, $p/CpCqp$ *$C7$—8,

8 $CCCpCqpCCCFCGvCwvuCCpCqpu$.

4′ $p/CCpCqpCCCFCGvCwvu$, $q/CpCqp$, r/u *$C8$—$C2'$—9,

9 $CCCpCqpCCCFCGvCwvuu$.

1′ $p/CCpCqpCCCFCGvCwvu$, q/u, r/Cxu *$C9$—$C2'$ p/u, q/x—10,

10 $CCCpCqpCCCFCGvCwvuCxu$.

From above prooflines, we can have the following theorem.

Theorem 7. *In the BCK-system, any axiom system is able to be changed into a single axiom being equivalent to the original.*

Proof. It is seen that if F is thesis, then $CCCpCqpCFvCwv$ is a thesis being equivalent to $CpCqp$ and F, where p, q, v, and w are not contained in F. Let $CCCpCqpCFvCwv$ be G, and let H be a thesis. $CCCpCqpCGCHvCwv$ implies G, H, and $CpCqp$, where G and H do not have p, q, v, and w. Therefore we have the above theorem.

From the above theorem, as example, we have the following single axiom of the classical two valued propositional calculus.

1) $CCCpCqpCCCCCrCstCCrsCrtCCCNrNsCsrvCwvuCxu$.

For the implicational calculus,

1) $CCCpCqpCCCCCrsCCstCrtCCCCrsrrvCwvuCxu$.

For the positive implicational calculus,

1) $CCCpCqpCCCCCrCrsCrsCCCrsCCstCrtvCwvuCxu$.

References

[1] Y. Arai: On axiom systems of propositional calculi. III. Proc. Japan Acad., **41**, 570-574 (1965).

[2] Y. Arai, K. Iséki, and S. Tanaka: Characterizations of *BCK*, *BCI*-algebras. Proc. Japan Acad., **42**, 105-107 (1966).

[3] Y. Imai and K. Iséki: On axiom systems of propositional calculi. I. Proc. Japan Acad., **41**, 436-439 (1965).

[4] ——: On axiom systems of propositional calculi. XIV. Proc. Japan Acad., **42**, 19-22 (1966).

[5] K. Iséki: An algebra related with a propositional calculus. Proc. Japan Acad., **42**, 26-29 (1966).

[6] K. Iséki and S. Tanaka: On axiom systems of propositional calculi. X. Proc. Japan Acad., **41**, 801-802 (1965).

[7] S. Tanaka: On axiom systems of propositional calculi. VI. Proc. Japan Acad., **41**, 663-666 (1965).

[8] ——: On axiom systems of propositional calculi. IX. Proc. Japan Acad., **41**, 798-800 (1965).