129. *On the Spectral Decomposition of Dissipative Operators*

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An operator A on a Hilbert space is called *dissipative* if its imaginary part is non-negative, i.e.,

$$Im(A)=\frac{1}{2i}(A-A^*)\geqq 0.$$

In the present paper, we shall concern ourselves with the spectral properties of dissipative operators with completely continuous imaginary part, and deduce the spectral decomposition of operators of this class in the case of real spectrum. Consequently, for completely continuous dissipative operators with real spectrum, the canonical reduction of Jordan type will be established.

For the sake of simplicity, we shall assume that our (complex) Hilbert space is separable. By an operator we always understand a bounded linear transformation on a Hilbert space. Let A be an operator on a Hilbert space H. Then we denote by $\sigma(A)$ the spectrum of A and by $P_\sigma(A)$ (resp. $C_\sigma(A)$, $R_\sigma(A)$) the point (resp. continuous, residual) spectrum of A. If $\sigma(A)=P_\sigma(A)$ (resp. $\sigma(A)=C_\sigma(A)$), we say that A has a pure point (resp. continuous) spectrum.

As in [5]-[6], throughout this paper we make use of the concept of von Neumann algebra. Let us recall the terminologies and the notations used in [5]-[6]. An operator A is said to be *primary* if the von Neumann algebra $R(A)$ generated by A is a factor (that is, its center consists of scalar multiples of the identity operator I), and a primary operator A is said to be *of type* I_n (resp. *of type* I_∞) if a factor $R(A)$ is of type I_n (resp. of type I_∞).

1. A typical example of completely continuous dissipative operators is the integral operator A of Volterra type on $L_2(0, 1)$ defined by

$$(Af)(x)=i\int_0^x f(t)dt.$$

It is well known that A is a primary operator with non-hermitian rank 1 and $\sigma(A)=C_\sigma(A)=\{0\}$, that is, A has a pure continuous spectrum. In [5]-[6], we have shown that a non-scalar primary operator with non-hermitian rank 1 has, in general, a pure continuous spectrum in case $\sigma(A)$ is real. This spectral property may be generalized to the dissipative case.

For simplicity we say that an operator A belongs to the class (iC) if its imaginary part $Im(A)$ is completely continuous. Now let us concentrate our attention on the dissipative operators of the class (iC). The first object is to show the spectral properties of our operators as follows:

Proposition 1. *Let A be a non-scalar primary dissipative operator of the class (iC). Then, for $\lambda \in \sigma(A)$;*

(1.1) *$\lambda \in P_\sigma(A)$ if and only if λ is non-real and $0 < \mathcal{I}m\lambda < \|Im(A)\|$,*

(1.2) *$\lambda \in C_\sigma(A)$ if and only if λ is real;*

and moreover we have

(1.3) *$R_\sigma(A)$ is empty.*

The proof of Proposition 1 is based on the following two lemmas which give a key to our observation.

Lemma 1. *Let A be a non-scalar primary dissipative operator on a Hilbert space H and let λ a complex number such that $\mathcal{I}m\lambda$ is not contained in the open interval $(0, \|Im(A)\|)$. Then the range of $A-\lambda I$ is dense in H. That is, for every real number λ, the range of $A-\lambda I$ is dense in H.*

Proof. We denote by $\mathcal{M}$ the range of $A-\lambda I$ and by $\mathcal{N}$ the orthogonal complement of $\overline{\mathcal{M}}$. Then, for every vector $\varphi \in \mathcal{N}$, we have

$$\begin{aligned}\langle 2iIm(A)\varphi, \varphi\rangle &= \langle[(A-\lambda I)-(A^*-\bar{\lambda}I)+2i\mathcal{I}m\lambda I]\varphi, \varphi\rangle \\ &= \langle(A-\lambda I)\varphi, \varphi\rangle - \langle\varphi, (A-\lambda I)\varphi\rangle + \langle 2i\mathcal{I}m\lambda\varphi, \varphi\rangle \\ &= \langle 2i\mathcal{I}m\lambda\varphi, \varphi\rangle.\end{aligned}$$

Thus

(1.4) $\langle(Im(A)-\mathcal{I}m\lambda I)\varphi, \varphi\rangle = 0$ for every vector $\varphi \in \mathcal{N}$.

Here, if $\mathcal{I}m\lambda \leqq 0$, $Im(A)-\mathcal{I}m\lambda I \geqq 0$, and if $\mathcal{I}m\lambda \geqq \|Im(A)\|$, we have $Im(A)-\mathcal{I}m\lambda I \leqq 0$. Therefore, in both cases the equality (1.4) implies that $(Im(A)-\mathcal{I}m\lambda I)\varphi = 0$. This means that $A\varphi = A^*\varphi + 2i\mathcal{I}m\lambda\varphi$ for every vector $\varphi \in \mathcal{N}$. Since $\mathcal{N}$ is clearly invariant by A^*, $\mathcal{N}$ reduces the operator A. In other wards, the projection P on $\mathcal{N}$ belongs to $R(A)'$. On the other hand, it is easy to see that P belongs to $R(A)$ (note that $\mathcal{M}$ reduces every operator of $R(A)'$). Accordingly, P must be I or 0 since A is primary. But obviously $P \neq I$, and hence $P=0$, that is to say, $\overline{\mathcal{M}} = H$.

Lemma 2. *Let A be a non-scalar primary dissipative operator. Then every point $\lambda \in P_\sigma(A)$ is non-real and $0 < \mathcal{I}m\lambda < \|Im(A)\|$. That is, A does not have a real proper value.*

Proof. Suppose that there exists $\lambda \in P_\sigma(A)$ such that $\mathcal{I}m\lambda \notin (0, \|Im(A)\|)$, and let $\mathcal{M}$ denote the proper subspace of A corresponding to λ (i.e., the proper subspace of $(-A)$ corresponding to $(-\lambda)$). Then it is easily seen that $\mathcal{M}$ is orthogonal to the range of $(-A^*)+\bar{\lambda}I$. However, since $(-A^*)$ is dissipative, it follows from Lemma 1 that the range of $(-A^*)+\bar{\lambda}I$ is dense in H if $\mathcal{I}m(-\bar{\lambda}) \notin (0, \|Im(-A^*)\|)$

(i.e., $\mathcal{J}m\lambda \notin (0, \|Im(A)\|)$). Thus $\mathcal{M}$ contains only zero vector, which is the contradiction.

We are now in position to prove the proposition. Indeed, our assertion follows immediately from the above lemmas.

Proof of Proposition 1. It is known in [1; Theorem 1] that for an operator A of the class $(i\mathcal{C})$,

(1.5) every non-real point $\lambda \in \sigma(A)$ belongs to $P_\sigma(A)$.

Therefore, (1.1) is directly obtained by the fact (1.5) and lemma 2. (1.2): It is clear from the fact (1.5) that $C_\sigma(A)$ consists of real numbers. If $\lambda \in \sigma(A)$ is real, (1.1) and lemma 1 imply $\lambda \in C_\sigma(A)$. Now (1.3) is a direct consequence of the fact (1.5) and (1.2).

2. In [5]-[6], we have shown that an operator A of the class $(i\mathcal{C})$ on a Hilbert space H is decomposed by a unique countable family of mutually orthogonal central projections P_0, $P_i (i \in I)$ in $R(A)$ into the form

(2.1) $$A = A_{P_0} \oplus \left(\textstyle\sum_{i \in I} \oplus A_{P_i}\right),$$

where the restriction A_{P_0} of A to P_0H is a self adjoint operator, the restriction A_{P_i} of A to $P_iH (i \in I)$ is a primary operator of type $I_\alpha (\alpha = n$ or $\infty)$ (which belongs to the class $(i\mathcal{C})$) and $P = \sum_{i \in I} P_i$ is the projection on the subspace generated by vectors of the form $A^n\varphi$ $(\varphi \in Im(A)H)$. Now let us consider a dissipative operator A of the class $(i\mathcal{C})$ whose spectrum is real. Then from the nature of the spectrum as mentioned in the preceding section and the result (2.1), we can see more exactly the structure of A which may be regarded as the spectral decomposition of A in a certain sense.

Theorem 1. *Let A be a dissipative operator of the class $(i\mathcal{C})$ with real spectrum on a Hilbert space H. Then A is decomposed by a unique countable family of mutually orthogonal central projections P_0, $P_i (i \in I)$ in $R(A)$ into the form*

$$A = A_{P_0} \oplus \left(\textstyle\sum_{i \in I} \oplus A_{P_i}\right),$$

where the restriction A_{P_0} of A to P_0H is a self adjoint operator and the restriction A_{P_i} of A to $P_iH (i \in I)$ is a primary operator of type I_∞ which has a pure continuous spectrum (and which is a dissipative operator of the class $(i\mathcal{C})$).

Proof. Since $\sigma(A)$ is real, in the decomposition (2.1) every $\sigma(A_{P_i})(i \in I)$ is also real. In this case, by the fact that PH is generated by vectors of the form $A^n\varphi(\varphi \in Im(A))$, each A_{P_i} does not admit to be a scalar operator. It follows from Proposition 1 that each operator A_{P_i} has a pure continuous spectrum. Consequently, it is impossible that A_{P_i} is of type $I_n (n = 1, 2, \cdots)$ since a primary operator of type I_n has necessarily a pure point spectrum by reason that the von Neumann algebra generated by it is $*$-isomorphic to the algebra of all operators on a n-dimensional Hilbert space.

Next let us restrict our attention to the completely continuous dissipative operators. Then Theorem 1 gives the complete spectral reduction for completely continuous dissipative operators with real spectrum.

Theorem 2.[1] *Let A be a completely continuous dissipative operator with real spectrum on a Hilbert space H. Then A is decomposed by a unique countable family of mutually orthogonal central projections P_0, $P_i (i \in I)$ in $R(A)$ into the form*

$$A = A_{P_0} \oplus (\textstyle\sum_{i \in I} \oplus A_{P_i}),$$

where the restriction A_{P_0} of A to $P_0 H$ is a self adjoint (completely continuous) operator and the restriction A_{P_i} of A to $P_i H (i \in I)$ is a quasi-nilpotent primary (completely continuous, dissipative) operator of type I_∞.

This fact means that the study of non-self adjoint operators of this class may be essentially reduced to that of quasi-nilpotent primary operators of type I_∞ (belonging to this same class). Moreover, for operators of this class, it gives a decomposition analogeous to the Jordan's canonical reduction for operators on a finite dimensional Hilbert space. Indeed, putting $S = A_{P_0} \oplus 0$ and $N = 0 \oplus (\sum_{i \in I} \oplus A_{P_i})$, we have the following

Theorem 3. *A completely continuous dissipative operator with real spectrum is expressible in the following form:*

$$A = S + N,$$

where S is a self adjoint (completely continuous) operator and N is a quasi-nilpotent (completely continuous, dissipative) operator which commutes with S.

This shows that our operator is a spectral operator in the sense of N. Dunford [3].

Corollary. *A completely continuous dissipative operator with real spectrum is a spectral operator.*

3. Related results. By what we have seen in the preceding section the following spectral properties of a dissipative operator (not necessarily primary) of the class (iC) can be deduced.

Proposition 2. *Let A be an arbitrary dissipative operator of the class (iC). Then*

(3.1) *if* $\lambda \in P_\sigma(A)$, $0 \leqq \mathcal{I}m\lambda \leqq \| Im(A) \|$,

(3.2) $R_\sigma(A)$ *is empty.*

Proof. (3.1): We may assume that $\lambda \in P_\sigma(A)$ is non-real. Then, in the decomposition (2.1) of A, λ belongs to $P_\sigma(A_{P_i})$ for some $i \in I$. In case A_{P_i} is a (non-real) scalar operator, $\mathcal{I}m\lambda = \| Im(A_{P_i}) \| = \| PIm(A) \| \leqq$

1) From this result it is readily obtained that a completely continuous dissipative operator with real spectrum belongs to the class (Ω) defined by M. S. Brodskiĭ [2].

$\|Im(A)\|$. If A_{P_i} is a non-scalar operator, $0<\mathcal{I}m\lambda<\|Im(A)\|$ by Proposition 1. (3.2): From Proposition 1 we can see that in (2.1) each of $R_\sigma(A_{P_0})$ and $R_\sigma(A_{P_i})(i\in I)$ is empty. Then the assertion is immediately concluded. In fact, $\lambda\in R_\sigma(A)$ implies that for each $k=0, i$, $(A-\lambda I)P_kH$ is dense in P_kH. Thus the range of $A-\lambda I$ is dense in H. This contradicts to the fact that λ is in $R_\sigma(A)$.

An operator A on a Hilbert space is called *completely non-self adjoint* if there does not exist a non-zero reducing subspace $\mathcal{M}$ of A such that the restriction $A_{\mathcal{M}}$ of A to $\mathcal{M}$ is a self adjoint operator. For an arbitrary operator A on a Hilbert space H, we denote by K the range of $Im(A)$ and by E the projection on the subspace $\bar{K}$, and we consider the subspace H_1 generated by vectors of the form $A^n\varphi$ $(\varphi\in K, n=0,1,2,\cdots)$ and denote by P the projection on H_1. We have shown in [6] that P is the central support of E in $R(A)$ (i.e., the minimal central projection containing E). From this fact it follows that the following conditions are equivalent to each other.

(3.3) *The central support of E in $R(A)$ is the identity operator.*

(3.4)[2] *H is generated by vectors of the form $A^n\varphi$ $(\varphi\in K, n=0,1,\cdots)$*

(3.5) *A is completely non-self adjoint.*

For the proof, we need only to verify the equivalence (3.4)$\rightleftarrows$(3.5). (3.4)$\to$(3.5): Assume that there exists a non-zero projection Q in $R(A)'$ such that A_Q is self adjoint. Then $EQ=QE=0$ (note that $E\in R(A)$). This implies $E\leqq I-Q$, and so $P\leqq I-Q<I$. (3.5)$\to$(3.4): If $P\neq I$, A_{I-P} is self adjoint. This is the contradiction since $I-P\neq 0$ belongs to $R(A)'$.

Now we have seen that an operator has the decomposition

$$A=A_{I-P}\oplus A_P, \tag{3.6}$$

where A_{I-P} is self adjoint and A_P is completely non-self adjoint. Moreover, as a direct consequence of the above statement, we can obtain that a primary operator which is not a real scalar operator is completely non-self adjoint. Recently C. Foias and B. Sz-Nagy [4] have pointed out that a completely non-self adjoint dissipative operator does not have a real proper value. Hence some of our results may be directly derived from this result.

As is well known, a self adjoint operator A can be reduced by a projection R in $R(A)$ to two cases: A_R has a pure point spectrum and A_{I-R} has a pure continuous spectrum. We know that a similar result holds for an operator with non-hermitian rank 1 in the case of real spectrum (see [6]). At the final step, we shall note here that this fact is valid for the dissipative case.

Proposition 3. *Let A be a dissipative operator whose spectrum*

2) An operator satisfying this condition is called *simple* in [1].

is real. Then A is decomposed by a central projection R in $R(A)$ into the form

$$A = A_R \oplus A_{I-R},$$

where A_R has a pure point spectrum and A_{I-R} has a pure continuous spectrum.

Proof. From (3.6) A has the form $A = A_{I-P} \oplus A_P$, where A_{I-P} is self adjoint and A_P is completely non-self adjoint. Moreover, as we know, A_{I-P} has the decomposition $A_{I-P} = A_R \oplus A_{(I-P)-R}$, where R is a central projection in $R(A)$ such that $R \leqq I-P$, A_R has a pure point spectrum and $A_{(I-P)-R}$ has a pure continuous spectrum. Since the spectrum of A_P is real, the result [4] mentioned above shows that A_P has a pure continuous spectrum, because $\lambda \in R_\sigma(A_P)$ yields that $-\lambda$ is a proper value of the completely non-self adjoint dissipative operator $(-A_P^*)$. Therefore, $A_{I-R} = A_{(I-P)-R} \oplus A_P$ has a pure continuous spectrum as seen in the proof of Proposition 2-(3.2).

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