225. *Integration on Locally Compact Spaces Generated by Positive Linear Functionals Defined on the Space of Continuous Functions with Compact Support and the Riesz Representation Theorem. II*[*)]

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(Comm. by Kinjirô KUNUGI, M.J.A., Nov. 12, 1966)

We shall use in this paper the same terminology and notation as in the paper [12].

The main result of § 3 is that the integral functional generated by the Baire volume or the Baire measure gives the smallest extension of the positive linear functional on C_0. It would be interesting to know if representations by means of Borel volumes or of Borel measures are also minimal. In § 5 is given a Baire type characterization of the space of Lebesgue-Bochner measurable functions generated by a positive linear functional defined on C_0. Every function f from the space $M_v(Y)$ of Lebesgue-Bochner measurable functions generated by the Borel or Baire volume v has the property that there exists a function g from the *second Baire class* $C(Y)$ such that $f(x)=g(x)$ v-almost everywhere.

The *classes* $C_n(Y)$ are defined by induction. The class $C_0(Y)$ consists of all continuous functions f with compact support from X into Y. The class $C_n(Y)$ consists of limits of sequences of functions from $C_{n-1}(Y)$.

Let $N(v, Y)$ denote the space of all functions f from the space X into the Banach space Y such that $f(x)=0$ v-almost everywhere.

It follows from the previous result that in every class of the quotient $L(v, Y)/N(v, Y)$ there exists a function such that $g \in C_2(Y)$. Thus if v_1 is a Borel volume and v is the Baire volume being its restriction to the Baire prering then the corresponding quotient spaces are isometrically isomorphic. Moreover the bilinear integral $\int u(f, d\mu)$ considered on the quotient spaces yields the same operator.

This result shows that if one identifies functions equal almost everywhere then the representation by means of Borel volumes or by means of Baire volumes yields essentially the same representation. Since the integration generated by Borel or Baire measure coincides with the integration generated by the corresponding Borel or Baire

*) This research was partially supported by N. S. F. Grant GP 2565.

volumes therefore the same is true for Borel or Baire measures. The density of the set $C_0(Y)$ in the spaces $L(v, Y)$, $M_v(Y)$ for the case of the Borel or Baire volume v is given in § 7.

All the above results have been generalized to a bigger class of volumes on locally compact spaces in [5].

In § 8, using the classical Riesz decomposition of a linear continuous functional on the space C of real-valued continuous functions on a compact space X, we obtain a representation which is a very close analog to the Riesz representation theorem. However, there is an essential difference: the integral used in this paper is defined on the space $L(|\mu|, R)$ which is larger than the space $S(|\mu|, R)$ of Stieltjes summable functions.

The class of spaces $L(v, Y)$ generated by volumes as can be seen from the results of [3] coincides with the class of spaces generated by complete sigma-finite measures. It follows from the results of [7], [8] that this class of spaces coincides with the class of spaces generated by any complete measure.

It is worthwhile to mention that representations of multilinear continuous operators from the product of the spaces $L(v, Y)$ into any Banach space have been found in [4].

§ 3. The smallest integral functional representing a Daniell functional defined on C_0. Let F be a family of integral functionals $\int_w$. We say that the functional $\int_{w_0} \in F$ is the smallest in the family F if $\int_{w_0} \subset \int_w$ for all $\int_w \in F$.

Theorem 4. *Let J be a Daniell functional defined on C_0 and v the Baire volume generated by J. Then $\int_v$ is the smallest integral functional $\int_w$ such that $J \subset \int_w$.*

Notice that in this theorem the volume w is considered to be defined on any prering subsets of X.

Let v_1 be the Borel volume and v be the Baire volume generated by a Daniell functional J. From the above theorem we get $\int_v \subset \int_{v_1}$. Since the Baire prering V is contained in the Borel prering V_1, we get $v \subset v_1$.

§ 4. A Baire type characterization of measurable functions generated by a Daniell functional on C_0. Let $L(w, Y)$ be the space of Lebesgue-Bochner summable functions and $M_w(Y)$ be the space of *Lebesgue-Bochner measurable functions* corresponding to the volume w.

Theorem 5. *Let w be either a Baire or a Borel volume. Then for every function $f \in M_w(Y)$ (or $L(w, Y)$) there exists a* *function* $g \in C_2(Y)$ $(g \in C_2(Y) \cap L(w, Y)$, *respectively) such that* $f(x) = g(x)$ *w-almost everywhere.*

This result has been generalized to a larger class of volumes in [5]. Compare also with the results of § 6 [3] and § 5 [2].

§ 5. Isometric isomorphism of the spaces $L(w, Y)/N(w, Y)$ for a Borel volume and its Baire subvolume. If v is a volume, denote by $L_v(Y)$ the quotient space $L(v, Y)/N(v, Y)$. The space of nullfunctions $N(v, Y)$ is defined as in the introduction.

For elements $f \in L_v(Y)$ define the operator $\int f dv = \int g dv$ and the norm $\| f \|_v = \| g \|_v$, where $g \in f$. This definition is correct, that is, it does not depend on the choice of the representative $g \in f$.

Let v be a Borel volume and w a Baire volume such that $w \subset v$.

Define an operator i mapping the space $L_v(Y)$ into the space $L_w(Y)$ defined as follows: if $f \in L_v(Y)$ and $g \in f \cap C_2(Y)$ then $h = i(f)$ is the class such that $g \in h$. It is easy to see that this operator is well defined. We have

Theorem 6. *The mapping i establishes an isometry and an isomorphism of the space $L_v(Y)$ with the space $L_w(Y)$ preserving the integral, that is the map i is linear and onto,* $\| f \|_v = \| i(f) \|_w$ *and* $\int f dv = \int i(f) dw$ *for all* $f \in L_v(Y)$.

This theorem permits us to identify the spaces $L_v(Y)$ and $L_w(Y)$. Notice that in general we have $L(w, Y) \subset L(v, Y)$. One can give examples of compact topological Hausdorff spaces in which not every compact set is G_δ and therefore the Baire prering does not coincide with the Borel prering and still $L(w, Y) = L(v, Y)$. It would be interesting to know whether the last equality always holds.

§ 6. Density of the set $C_0(Y)$ in the spaces $L(v, Y)$ and $M_v(Y)$.

Theorem 7. *Let v be either a Borel or Baire volume. Then for every function* $f \in L(v, Y)$ $(f \in M_v(Y))$ *there exists a sequence of functions f_n from the set $C_0(Y)$ such that* $\| f_n - f \|_v \to 0$ $(f_n(x) \to f(x)$ *v-almost everywhere, respectively).*

From this theorem one can get that the classes containing elements of the set $C_0(Y)$ are dense in the space $L_v(Y)$.

§ 7. Representation of linear continuous functionals on the space C. Let (X, V, v) be a volume space. Take the space $L(v, Y)$ of Lebesgue-Bochner summable functions and denote by $M(v, Y)$ the space of all finitely additive functions from the prering V into a Banach space Z such that there exists a constant c such that

$$| \mu(A) | \leq c v(A) \qquad \text{for all } A \in V.$$

Let as before $U = L(Y, Z; W)$. By $(v)\int u(f, d\mu)$ we shall denote the operator developed in [1] and considered for $u \in U$, $f \in L(v, Y)$, $\mu \in M(v, Z)$.

If the bilinear operator is given by the formula $u(y, z) = zy$ for $y \in Y, z \in R$ then the corresponding operator will be denoted by $(v)\int f d\mu$ for $f \in L(v, Y)$, $\mu \in M(v, R)$.

From linearity of the operator $(v)\int u(f, d\mu)$ in the variable $\mu \in M(v, Z)$ we get linearity of the operator $(v)\int f d\mu$ in that variable.

Theorem 8. *Let* $\mu \in M(v, R)$ *then* $L(v, Y) \subset L(|\mu|, Y)$ and $\int f d\mu = (v)\int f d\mu$ for $f \in L(v, Y)$.

For related results to this theorem see [9].

Now let X be a compact Hausdorff space and C be the space of real-valued continuous functions on it. Notice that the norm in the space is defined by $\|f\| = \sup\{|f(x)| : x \in X\}$.

Denote by P the set of all functions $f \in C(X)$ such that $f(x) > 0$ for all $x \in X$. Notice that $C(X) = P - P$.

Let $K = \{f \in C(X) : \|f\| \leq 1\}$. Take any linear continuous functional h on the space $C(X)$. Define

$$h_1(f) = \sup\{h(fg) : g \in K\}$$

for $f \in P$. Since if $f_1, f_2 \in P$ then $f_1K = (f_1 + f_2)K$, the functional h_1 is additive on P.

It is easy to see that if $t > 0$ and $f \in P$ then $h_1(tf) = th_1(f)$.

Now if $f \in C(X)$ and $f_i \in P$ are such that $f = f_1 - f_2$ then define $h_1(f) = h_1(f_1) - h_1(f_2)$. One can prove that the functional h_1 is well defined and linear.

Notice that $0 \leq h_1(f) \leq \|h\| \|f\|$ for $f \in P$. Thus if $f = f_1 - f_2$ where $f_1 = f^+ + \|f\|$, $f_2 = f^- + \|f\|$ then $|h_1(f)| \leq 4\|h\| \|f\|$. This shows that the functional h_1 is continuous.

Now from the inequality $h_1(f) \geq 0$ for $f \in P$ and from continuity we get $h_1(f) \geq 0$ if $f(x) \geq 0$ for all $x \in X$. This proves that the functional is positive.

Again from the inequality $h(f) \leq h_1(f)$ for $f \in P$ we get $h(f) \leq h_1(f)$ if $f(x) \geq 0$ for all $x \in X$. Thus the functional $h_2 = h_1 - h$ is linear, continuous and positive. In this way we have $h = h_1 - h_2$.

Since the space X is compact therefore the spaces C_0, C coincide. Thus every positive linear functional can be represented by means of an integral with respect to a Baire volume, according to § 1. That is, we have $h_i = \int f dv_i$ for $f \in C$.

Put $\mu = v_1 - v_2$, $v = v_1 + v_2$. We see that $v_1, v_2, \mu \in M(v, R)$.

Now if $f \in C$ then

$$h(f)=\int f dv_1-\int f dv_2=(v)\int f dv_1-(v)\int f dv_2$$
$$=(v)\int f d\mu=\int f d\mu.$$

From density of the space C we easily get

Theorem 9. *For every linear continuous functional on the space C there exists one and only one volume μ from the Baire prering into R such that*

$$h(f)=\int f d\mu \quad \textit{for all } f\in C \textit{ and } \|h\|=|\mu|(X).$$

In exactly the same way one gets representations of the functional on the Borel prering and on the Borel and Baire rings.

This result is a very close analog of the Riesz representation theorem as mentioned in the introduction. It is worthwhile to mention that it has been generalized to the case of the space of continuous functions f from a locally compact Hausdorff space X into any locally convex space Y. The space Y does not have to be Hausdorff or complete. This result will appear in [6].

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