66. *Relations between Complete Integral Seminorms and Complete Volumes*

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Let μ be a measure on a σ-ring M. Denote by $v=t\mu$ the function defined by the formula: $v(A)=\mu(A)$ for $A\in V$, where

$$V=\{A\in M:\mu(A)<\infty\}.$$

It is easy to see that the family V is a prering and the function v is a volume. This volume will be called the finite part of the measure μ. If one follows carefully any construction of the space $L_\mu(Y)$ of Lebesgue-Bochner summable functions generated by the measure μ one notices that essentially one needs only the finite part of the measure.

Further observation yields that one needs actually only a functional J which we call a complete integral seminorm. This functional is given by the formula

$$Jf=\int fd\mu\ (f\in L_\mu^+),$$

where L_μ^+ consists of all finite-valued μ-summable nonnegative functions. In this paper we shall find inner characterizations of complete integral seminorms.

If f, g are two real valued functions then by $f\cap g, f\cup g, f\cap 1$ we shall understand the functions $(f\cap g)(x)=inf\{f(x), g(x)\}$, $(f\cup g)(x)=sup\{f(x), g(x)\}$, $(f\cap 1)(x)=inf\{f(x), 1\}$ for all $x\in X$.

We shall write $f\leq g$ if $f(x)\leq g(x)$ for all $x\in X$. In a similar way we define the relation $f\geq g$.

A sequence f_n is called increasing (decreasing) if the condition $n\leq m$ implies $f_n\leq f_m$ ($f_n\geq f_m$, respectively).

A nonnegative functional J is called an integral seminorm over the space X if its domain J^+ consists of functions from X into $R^+=\langle 0,\infty)$ and the following three conditions are satisfied:

(1) If $t_1, t_2\in R^+$ and $f_1, f_2\in J^+$ then $t_1f_1+t_2f_2\in J^+$ and

$$J(t_1f_1+t_2f_2)=t_1Jf_1+t_2Jf_2.$$

(2) If $f, g\in J^+$ then $f\cup g\in J^+$ and $f\cap 1\in J^+$.

(3) If $f\leq g$ and $f, g\in J^+$ then $g-f\in J^+$.

The integral seminorm is called *upper complete* if, for every increasing sequence $f_n\in J^+$, converging at every point of the space to a finite-valued function f, for which the sequence of numbers Jf_n is bounded, we have $f\in J^+$ and $Jf_n\to Jf$.

An integral seminorm is called *complete* if in addition it satisifies the following condition: If $0 \leq g \leq f \in J^+$ and $Jf=0$ then $g \in J^+$.

Example 1. Let M be a σ-ring of subsets of a space X. Let μ be a measure on M. Let J^+ consist of all μ-summable finite-valued nonnegative functions and let

$$Jf=\int f d\mu \text{ for } f \in J^+.$$

Then J is an upper complete integral seminorm.

If the measure μ is complete that is if it has the following property: $A \subset B \in M$ and $\mu(B)=0$ implies $A \in M$, then the functional J is a complete integral seminorm. Since every measure admits a complete extension, see for example Halmos [14], therefore every measure generates a complete integral seminorm.

It is interesting to notice that the construction of the integral developed in Dunford and Schwartz [15] has the properties that the measure μ generates the same integral seminorm as its completion μ_c([15] p. 147), that is we have the see L_μ^+ of all non-negative finite-valued summable functions generated by μ coincides with the set $L_{\mu_c}^+$ and we have $\int f d\mu=\int f d\mu_c$ for all $f \in L_\mu^+$.

For other methods to generate integral seminorms see [7], [13], [16], [17].

If v is a volume on a prering V of a space X then the triple (X, V, v) is called a *volume space*.

If F is a family of real valued functions on X then by F^+ we shall denote the family of all non-negative functions from F.

Example 2. Let (X, V, v) be a volume space and $L(v, R)$ be the corresponding space of summable functions (see [1]). Put

$$J^+=L^+(v, R) \text{ and } Jf=\int f dv \text{ for } f \in J^+.$$

It follows from Theorems 1 and 2, [1] that the functional J is a complete integral seminorm.

Denote by i the operator prescribing the integral seminorm J to the measure μ as in Example 1, that is $J=i\mu$.

We shall use the same symbol to denote the integral seminorms generated by a volume v as in Example 2. Thus to indicate that the functional is generated by the volume v we shall write $J=iv$.

A volume v with the domain V is called *upper complete* if the following two conditions are satisfied:

(1) The family V is a ring that is in addition to axioms of a prering it satisfies the following condition: if $A, B \in V$ then $A \cup B \in V$.

(2) For every sequence of increasing sets $A_n \in V$ such that the sequence $v(A_n)$ of numbers is bounded we have $A=\bigcup_n A_n \in V$.

If in addition the volume satisfies the following condition: $A \subset B \in V$ and $v(B)=0$ implies $A \in V$, then the volume v is called complete.

Denote by g the operator mapping an integral seminorm J into the set function $v=gJ$ defined by the conditions

$$V=\{A \in X : \chi_A \in J^+\}$$

and $$v(A)=J\chi_A \text{ for all } A \in V.$$

Theorem 1. *If J is a complete integral seminorm then $v=gJ$ is a complete volume such that $J=iv$, that is*

$$J^+=L^+(v, R) \text{ and } Jf=\int f dv \text{ for all } f \in J^+.$$

The proof is based on the following lemmas and on results of [1].

Lemma 1. *If $f_1, f_2 \in J^+$ and $f_1 \leq f_2$ then $Jf_1 \leq Jf_2$.*

Define the following family of functions

$$N^+=\{f \in J^+ : Jf=0\}.$$

This family of functions will be called the family of null-functions corresponding to the integral seminorm J.

Lemma 2. *If J is a complete integral seminorm and $0 \leq g \leq f$ and $f \in N^+$ then $g \in N^+$.*

Denote by N the family of all sets $A \subset X$ such that $\chi_A \in N^+$. This family will be called the family of null-sets generated by the integral seminorm.

A family F of subsets of a space X is called a sigma-ideal if the following two conditions are satisfied:

(1) *If $A \subset B \in F$ then $A \in F$,*

(2) *If $A_n \in F$ is a sequence of sets then $\bigcup_n A_n \in F$.*

Lemma 3. *The family N forms a sigma-ideal of sets.*

Lemma 4. *Let $f \in J^+$. Then the following conditions are equivalent: $f \in N^+$ and $\{x \in X: f(x) \neq 0\} \in N$.*

Lemma 5. *Let $f_n \in J^+$ be a decreasing sequence convergent at every point of X to a function f. Then $f \in J^+$ and $Jf_n \to Jf$.*

Lemma 6. *If $f_1, f_2 \in J^+$ then $f_1 \cap f_2 \in J^+$.*

Denote by t the operator mapping a measure μ on a semi-ring M of subsets of X into its finite part $v=t\mu$. That is into a set function defined on

$$V=\{A \in M : \mu(A) < \infty\}$$

by the formula $v(A)=\mu(A)$ for all $A \in V$.

Notice that the function $v=t\mu$ is an upper complete volume.

Theorem 2. *Let μ be a complete measure and $J=i\mu$. Then the finite part of μ coincides with the volume $v=gJ$, that is $t\mu=gJ$.*

As an immediate consequence of Theorem 2 we get the corollary.

Corollary 1. *Let μ_1, μ_2 be complete measures defined on some* *sigma-rings of a space X. Then the measures generate the same complete integral seminorm, that is $J=i\mu_1=i\mu_2$, if and only if, the measures have the same finite part, that is*

$$v=t\mu_1=t\mu_2.$$

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