194. *On Free Abelian m-Groups. II*

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In the second part of this article, the notion of free abelian m-group will be introduced and their properties are given.

Definition. An m-group $(M, [\ ])$ will be called *abelian* if and only if $[x_1x_2\cdots x_m]=[x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(m)}]$ for every permutation σ of $1, 2, \cdots, m$ and each $x_1, x_2, \cdots, x_m \in M$.

Definition. An abelian m-group F is said to be *free* on X if and only if for a one-to-one function $i: X\to F$, an abelian m-group M, every function $g: X\to M$ has a unique homomorphism extension $h: F\to M$ that makes the following diagram commutative

$$\begin{tikzcd} X \arrow[d, "i"'] \arrow[dr, "g"] & \\ F \arrow[r, "h"'] & M, \end{tikzcd}$$

that is to say, $h\circ i=g$.

Consider the (restricted) direct sum $\sum_{x\in X} Z_x$ of copies $Z_x=Z$ of the additive group of the integers Z. Recall that $f\in\sum_{x\in X} Z_x$ if and only if $f(x)=0$ for all x except for a finite number $x_1, \cdots, x_n$ of elements of X. For each $x\in X$, if $|x|$ denotes the member of $\sum_{x\in X} Z_x$ such that $|x|(x)=1$ but otherwise is zero, then

$$f=f(x_1)|x_1|+\cdots+f(x_n)|x_n|.$$

Set

$$F=\left\{f\in\sum_{x\in X} Z_x \mid \sum_{x\in X} f(x)\equiv 1(\text{mod } m-1)\right\}.$$

Under the m-ary operation defined by

$$[f_1f_2\cdots f_m]=f_1+f_2+\cdots+f_m,$$

where f_i+f_j is the function such that $(f_i+f_j)(x)=f_i(x)+f_j(x)$, the system $(F, [\ ])$ is obviously an abelian m-group. Note that every integer $f(x_i)$ is a sum of a minimal number of the integers $\langle 0\rangle$ and $\langle -1\rangle$. This minimal sum is unique except for ordering. This means that every element $f=f(x_1)|x_1|+\cdots+f(x_n)|x_n|$ of F possesses a unique factorization (up to ordering or arrangement)

$$\begin{aligned} f &= |x_1|^{\langle e_{11}\rangle}+|x_1|^{\langle e_{12}\rangle}+\cdots+|x_1|^{\langle e_{1r_1}\rangle}+\cdots \\ &\quad +|x_n|^{\langle e_{n1}\rangle}+|x_n|^{\langle e_{n2}\rangle}+\cdots+|x_n|^{\langle e_{nr_n}\rangle} \\ &= [|x_1|^{\langle e_{11}\rangle}|x_1|^{\langle e_{12}\rangle}\cdots|x_1|^{\langle e_{1r_1}\rangle}\cdots|x_n|^{\langle e_{n1}\rangle}\cdots|x_n|^{\langle e_{nr_n}\rangle}], \end{aligned}$$

where $e_{ij}=0$ or -1 and $\sum_{i=1}^{n}\sum_{j=1}^{r_i}\langle e_{ij}\rangle\equiv 1$ (mod $m-1$). Observe that

$|x|^{\langle -1\rangle}$ is the element of $\sum_{x\in X} Z_x$ such that $|x|^{\langle -1\rangle}(x)=-m+2$ but otherwise is 0. Note also that it is the unique element such that $(|x|, |x|, \cdots, |x|, |x|^{\langle -1\rangle})$ and $(|x|^{\langle -1\rangle}, |x|, \cdots, |x|)$ are $(m-1)$-adic identies or, in other words, $|x|+|x|+\cdots+|x|+|x|^{\langle -1\rangle}$ is the function which is identically 0.

Theorem 4. *The abelian m-group $(F, [\quad])$ defined above is free* on X.

Proof. For an arbitary m-group M and $g: X\to M$ define $h: F\to M$ by $h([|x_1|^{\langle e_1\rangle}|x_2|^{\langle e_2\rangle}\cdots|x_k|^{\langle e_k\rangle}])=[g(x_1)^{\langle e_1\rangle}g(x_2)^{\langle e_2\rangle}\cdots g(x_k)^{\langle e_k\rangle}]$ where $e_i=0$ or -1, $\sum_{i=1}^{k}\langle e_i\rangle\equiv 1 \pmod{m-1}$, and $g(x)^{\langle -1\rangle}$ is the unique element of M such that $(g(x), g(x), \cdots, g(x), g(x)^{\langle -1\rangle})$ and $(g(x)^{\langle -1\rangle}, g(x), \cdots, g(x))$ are $(m-1)$-adic identities of M. It readily follows that h is an m-group homomorphism, i.e. $h([f_1f_2\cdots f_m])=[h(f_1)h(f_2)\cdots h(f_m)]$ for all $f_1, f_2, \cdots, f_m\in F$. If $i: X\to F$ is defined naturally by $i(x)=|x|$, then $(h\circ i)(x)=h(i(x))=h(|x|)=g(x)$. Note that if $h\circ i=g=h'\circ i$, then $h'=h$.

Corollary 5. *Every m-group M is the quotient m-group of a free m-group.*

Proof. Let $i: M\to F$ be that $i(x)=|x|$ for all $x\in M$, where F is a free m-group on M, and $1: M\to M$ be the identity function. Then, by Theorem 4, there exists uniquely a homomorphism $h: F\to M$ such that $h\circ i=1$. Hence h is onto M and $F/h\circ h^{-1}$ is isomorphic to M.

Corollary 6. *If F_1 is an abelian m-group free on X_1, F_2 is an abelian m-group free on X_2, and X_1 and X_2 have the same number of elements, then F_1 and F_2 are isomorphic.*

Proof. In the following diagram

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "e"] \arrow[d, "i_1"'] \arrow[dr] & X_2 \arrow[l, shift left, "e^{-1}"] \arrow[d, "i_2"] \arrow[dl] \\
F_1 \arrow[r, shift left, "h_1"] & F_2 \arrow[l, shift left, "h_2"]
\end{tikzcd}$$

let $e: X_1\to X_2$ be one-to-one and onto and $i_k: X_k\to F_k (k=1, 2)$ be one-to-one. Then, by Theorem 4, there exist uniquely homomorphisms $h_1: F_1\to F_2$ and $h_2: F_2\to F_1$ such that $h_1\circ i_1=i_2\circ e$ and $h_2\circ i_2=i_1\circ e^{-1}$. Then, $h_2\circ h_1\circ i_1=h_2\circ i_2\circ e=i_1\circ e^{-1}\circ e=i_1$ and hence $h_2\circ h_1=1$. Similarly, $h_1\circ h_2\circ i_2=h_1\circ i_1\circ e^{-1}=i_2\circ e\circ e^{-1}=i_2$ and hence $h_1\circ h_2=1$. Whence h_1 and h_2 are homomorphisms that are inverses of each other.

Corollary 7. *The free abelian m-group F on X is a coset of* *the free abelian (2-group) group* $\sum_{x \in X} Z_x$ *on* X *under the (normal) subgroup* N *of all* $f \in F$ *such that*

$$\sum_{x \in X} f(x) \equiv 0 \pmod{m-1}.$$

Proof. Obviously, N is closed under $+$ and $-$ and therefore a subgroup of $\sum_{x \in X} Z_x$. Moreover, observe that $F = N + |x|$ for any $x \in X$.

Corollary 8. *The free abelian m-group on a singleton* $\{x\}$ *(i.e. the infinite cyclic m-group* $(|x|)$*) is isomorphic to the m-group* $(Z, [\quad])$ *of integers under the operation*

$$[n_1 n_2 \cdots n_m] = n_1 + n_2 + \cdots + n_m + 1$$

for all $n_1, n_2, \cdots, n_m \in Z$.

Proof. By the preceding Corollary 7, the infinite cyclic m-group is a coset of the additive group Z of integers by the (normal) subgroup Z_{m-1} of all multiples of $m-1$. In fact, $(|x|) = 1 + Z_{m-1}$. The function $h\colon (|x|) \to Z$ such that $h(\langle n \rangle) = n$ is clearly one-to-one and onto and also satisfies the relation

$$\begin{aligned} h([\langle n_1 \rangle \langle n_2 \rangle \cdots \langle n_m \rangle]) &= h(\langle n_1 + n_2 + \cdots + n_m + 1 \rangle) \\ &= n_1 + n_2 + \cdots + n_m + 1 = [n_1 n_2 \cdots n_m] = [h(n_1) h(n_2) \cdots h(n_m)] \end{aligned}$$

for all $n_1, n_2, \cdots, n_m \in Z$.

Now, consider any two abelian m-groups M and N. Let F be the free abelian m-group on the cartesian product $M \times N$. As we have seen before, an arbitrary element of F may, up to ordering, be uniquely representable in the form

$$\begin{aligned} &[|(x_1, y_1)|^{\langle e_1 \rangle} |(x_2, y_2)|^{\langle e_2 \rangle} \cdots |(x_k, y_k)|^{\langle e_k \rangle}] \\ &\quad = |(x_1, y_1)|^{\langle e_1 \rangle} + |(x_2, y_2)|^{\langle e_2 \rangle} + \cdots + |(x_k, y_k)|^{\langle e_k \rangle} \end{aligned}$$

where $e_i = 0$ or -1 and $\sum_{i=1}^{k} \langle e_i \rangle \equiv 1 \pmod{m-1}$. Let R be the symmetric relation on F that contains all pairs

$(|(x^{\langle n \rangle}, y)|, |(x, y^{\langle n \rangle})|)$, $(|([x_1 x_2 \cdots x_m], y)|, [|(x_1, y)| \, |(x_2, y)| \cdots |(x_m, y)|])$, and

$$(|(x, [y_1 y_2 \cdots y_m])|, [|(x, y_1)| \, |(x, y_2)| \cdots |(x, y_m)|])$$

for all $x, x_1, \cdots, x_m \in M$, $y, y_1, \cdots, y_m \in N$, and $n \in Z$. Let θ be the least congruence (relation) on F containing R. Note that $(v, w) \in \theta$ if and only if $v = [v_1 v_2 \cdots v_{\langle k \rangle}]$, $w = [w_1 w_2 \cdots w_{\langle k \rangle}]$, and for each $i = 1, 2, \cdots, \langle k \rangle$, there exists $u_{i1}, u_{i2}, \cdots, u_{ir_i} \in F$ such that $v_i = u_{i1}$, $w_i = u_{ir_i}$ and $(u_{ij}, u_{ij+1}) \in R$.

By the Post Coset Theorem, M is a coset of an abelian group $A = M \cup M^2 \cup \cdots \cup M^{m-1}$ and N of $B = N \cup N^2 \cup \cdots \cup N^{m-1}$. Recall that if F^* is the abelian group free on $A \times B$ and θ^* is the smallest congruence (relation) on F^* containing all ordered pairs

$$\begin{aligned} &(|(x_1 + x_2, y)|, |(x_1, y)| + |(x_2, y)|), \\ &\quad (|(x, y_1 + y_2)|, |(x, y_1)| + |(x, y_2)|), (|(-x, y)|, |(x, -y)|) \end{aligned}$$

for each $x, x_1, x_2 \in A$ and $y, y_1, y_2 \in B$, then F^*/θ^* is the tensor product $A \otimes B$ of the abelian groups A and B. Since

$$(|([x_1x_2 \cdots x_m], y)|, [|(x_1, y)||(x_1, y)| \cdots |(x_m, y)|])$$
$$=(|(x_1+x_2+ \cdots +x_m, y)|, |(x_1, y)|+|(x_2, y)|+ \cdots +|(x_m, y)|) \in \theta^*,$$
$$(|(x, [y_1y_2 \cdots y_m])|, [|(x, y_1)||(x, y_2)| \cdots |(x, y_m)|])$$
$$=(|(x, y_1+y_2+ \cdots +y_m)|, |(x, y_1)|+|(x, y_2)|+ \cdots +|(x, y_m)|) \in \theta^*,$$

and

$$(|(x^{\langle n \rangle}, y)|, |(x, y^{\langle n \rangle})|)=(|(\langle n \rangle x, y)|, |(x, \langle n \rangle y)|) \in \theta^*,$$

then $R \subseteqq \theta^*$ and hence $\theta \subseteqq \theta^*$.