36. *On a Riemann Definition of the Stochastic Integral.* I

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(Comm. by Kinjirô KUNUGI, M. J. A., Feb. 12, 1970)

§ 1. Introduction. Let $\{\beta_t(\omega); t \in [0, T]\}$ be a one-dimensional R^1-valued Brownien motion process, and let N_t^s be the smallest σ-algebra generated by $\{\beta_\tau(\omega); s \leqslant \tau \leqslant t\}$. Let S be the class of functions $f_t(\omega)$ on $[0, T] \times \Omega$ satisfying the following conditions.

S.1) $f_t(\omega)$ is $B_{[s,t]} \times N_t^s$-measurable for every $t \in [s, T]$, where $B_{[s,t]}$ is the Borel field on the interval $[s, t]$.

S.2) $M\left(\int_s^t f_\tau^2(\omega)\, d\tau\right) < +\infty$ for $0 \leqslant s \leqslant t \leqslant T$,

where $M(\cdot)$ denotes the expectation.

We will call a family of partitions $\Delta^{(n)}$ "canonical" if $\max(t_{i+1}^{(n)} - t_i^{(n)}) \cdot n$ tends to a constant as $n \to \infty$, where $\Delta^{(n)} = \{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = T\}$. Let us consider a following Riemann sum of a function $f_t(\omega)$ which belongs to the class S.

$$S_n(f)(\omega) = \sum_{i=0}^{n} f_{t_{i+1}^{(n)} + k(t_{i+1}^{(n)} - t_i^{(n)})}(\omega)(\beta_{t_{i+1}^{(n)}}(\omega) - \beta_{t_i^{(n)}}(\omega)) \tag{1}$$

where $0 \leqslant k \leqslant 1$.

Now our aim is to investigate conditions for the existence of the l.i.m. $S_n(f)(\omega)$. As for this problem, it is well known that if the interpolation ratio k is fixed to zero the limit of the series $S_n(f)(\omega)$ exists and equals to the Ito's stochastic integral $\int_0^T f_t(\omega) d^0\beta_t(\omega)$,*) while if the interporlation ratios are taken arbitrarily in each interval $(t_i^{(n)}, t_{i+1}^{(n)})$ it may fail to converge.

We will concern only with the series (1), where the ratios of interporlation are fixed to a certain constant $k (0 \leqslant k \leqslant 1)$ through all the intervals $(t_i^{(n)}, t_{i+1}^{(n)})$. Now the difficulty of this problem lies in the fact that the each random variables $f_{t_i^{(n)} + k(t_{i+1}^{(n)} - t_i^{(n)})}(\omega)$ $(i = 0, 1, \cdots, n-1)$ are not independent of the corresponding increments $\beta_{t_{i+1}^{(n)}}(\omega) - \beta_{t_i^{(n)}}(\omega)$ $(i = 0, 1, \cdots, n-1)$. So it seems to be necessary to put on the functions $f_t(\omega)$ one more condition which describes the way of dependence of $f_t(\omega)$ on the process $\beta_t(\omega)$.

To express this condition we will introduce a notion of β-differentiability of the functions $f_t(\omega)$ in § 2. With the help of this

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*) To distinguish the Ito's integral from the other types of integrals the notation $d^0\beta_t$ will be used.

notion we will be able to get rid of the difficulty. In § 3 we will give an answer to the problem in the Theorem 5 which is the main theorem of this paper. Theorem 5 represents the relation between the limit of the series (1), say $I_k(f)(\omega)$ and the Ito's stochastic integral of the function $f_t(\omega)$.

While the problem stated above is a interesting one, thus derived integral $I_k(f)(\omega)$, called the stochastic integral of index k, has its own property, that is, it is considered as somewhat generalized form of the stochastic integral. In fact, if the ratio k is equal to zero, it coincides with the Ito's integral and if k is a fixed number other than zero, the integral $I_k(f)(\omega)$ becomes a generalization of the stochastic integral introduced by R. L. Stratonovich (2) in 1964. As for the relation between the integral $I_k(f)(\omega)$ and the Stratonovich's one will be mentioned in § 3. Among the family of the integrals $I_k(f)(\omega)(0\leqslant k\leqslant 1)$, $I_{\frac{1}{2}}(f)(\omega)$ is important from the viewpoint of applied mathematics, because the integral $I_{\frac{1}{2}}(f)(\omega)$ becomes the limit of a certain sequence of Stieltjes integrals. As for this fact, the detailes are mentioned in § 4. The method developed in this paper will be applied in a forthcoming paper (S. Ogawa (3)).

§ 2. β-derivatives of the stochastic processes.

In order to study the convergence of the Riemann sum $S_n(f)$, it seems to be necessary to introduce a notion which describes the way of dependence of the integrand $f_t(\omega)$ on the process $\beta_t(\omega)$.

Definition. Let $X_t(\omega)$ be an R^1-valued, $B_{[0,t]}\times N_t^0$-measurable stochastic process defined on $[0, T]\times\Omega$. If for some n there exists a stochastic process $\zeta_t(\omega)((t,\omega)\in[0,T]\times\Omega)$ satisfying following conditions $\zeta.1)\sim\zeta.3)$, we shall call this the right β-derivative in the L_{2n}-sense of the process $X_t(\omega)$ with respect to the process $\beta_t(\omega)$ (or in short, the β^+-derivative of $X_t(\omega)$ in the L_{2n}-sense) and denote it as follows

$$\frac{\partial^+}{\partial^+\beta_t}X_t(\omega)=\zeta_t(\omega) \tag{2}$$

$\zeta.1)$ $\zeta_t(\omega)$ is a $B_{[0,t]}\times N_t^0$-measurable process.

$\zeta.2)$ $M(|\zeta_t(\omega)|^{2n})<+\infty$ for $t\in[0,T]$,

$\zeta.3)$
$$\lim_{s\downarrow t} M\left[\frac{1}{\sqrt{s-t}}\{X_s(\omega)-X_t(\omega)-\zeta_t(\omega)(\beta_s(\omega)-\beta_t(\omega))\}\right]^{2n}=0 \quad \text{for } 0\leqslant t\leqslant s\leqslant T$$

For a given R^1-valued, $B_{[t,T]}\times N_T^t$-measurable stochastic process $\tilde{X}_t(\omega)$, $((t,\omega)\in[0,T]\times\Omega)$, its left derivative with respect to the process $\beta_t(\omega)$ will be defined similarly.

If for stochastic process $\tilde{X}_t(\omega)$, there exists a stochastic process $\tilde{\zeta}_t(\omega)$ satisfying the following conditions $\tilde{\zeta}.1)\sim\tilde{\zeta}.3)$, we shall say that $\tilde{X}_t(\omega)$ has a left β-derivative in the L_{2n}-sense on $[0, T]$, and write

$$(2)' \qquad \frac{\partial^-}{\partial^-\beta_t}\tilde{x}_t(\omega)=\tilde{\zeta}_t(\omega)$$

$\tilde{\zeta}.1)$ $\tilde{\zeta}_t(\omega)$ is a $B_{[t,T]}\times N_T^t$-measurable process

$\tilde{\zeta}.2)$ same as $\zeta.2)$

$\tilde{\zeta}.3)$ $$\lim_{s\uparrow t} M\left[\frac{1}{\sqrt{t-s}}\{\tilde{x}_t(\omega)-\tilde{x}_s(\omega)-\tilde{\zeta}_t(\omega)(\beta_t(\omega)-\beta_s(\omega))\}\right]^{2n}=0$$

for $0\leq s\leq t\leq T$

It must be noticed that it does not make sense to discuss about the both β^+, β^--derivatives for a stochastic process at the same time. Though the notion of the β^--deribatives and some lemmas concerning it presented later are not necessary for the main purpose of this paper, they will be presented for their own interest.

The next theorem assures that these derivatives are well defined.

Theorem 1. *If a $B_{[0,t]}\times N_t^0(B_{[t,T]}\times N_T^t)$-measurable stochastic process $X_t(\omega)$ is $\beta^+(\beta^-)$-differentiable in the L_{2n}-sense then its derivative $\frac{\partial^+}{\partial^+\beta_t}X_t(\omega)\left(\frac{\partial^-}{\partial^-\beta_s}X_t(\omega)\right)$ is uniquely determined up to stochastic equivalence.*

Now let us show some examples of the β-derivatives.

Example 1. Let $\xi_t(\omega)$ be the diffusion process, determined by the following Ito's stochastic integral equation.

$$(3) \qquad \xi_t(\omega)-x=\int_0^t a(\tau,\xi_\tau(\omega))d\tau+\int_0^t b(\tau,\xi_\tau(\omega))\,d^0\beta_\tau(\omega),$$

where $a(t,x)$, $b(t,x)$ are bounded and continuous in t and Lipschitz continuous in x on $[0,T]\times R^1$.

Then this diffusion process $\xi_t(\omega)$ is β^+-differentiable in the L_{2n}-sense and its β^+-derivative is as follows

$$(4) \qquad \frac{\partial^+}{\partial^+\beta_t}\xi_t(\omega)=b(t,\xi_t(\omega)),$$

The notation $\frac{\partial}{\partial\beta_t}$, for the β-derivative is based on this fact.

Example 2. Let $\xi_s^{(t,x)}(\omega)$ be the diffusion process determined by the following stochastic integral equation.

$$(5) \qquad \xi_s^{(t,x)}(\omega)-x=\int_t^s a(\tau,\xi_\tau^{(t,x)}(\omega))d\tau+\int_t^s b(\tau,\xi_\tau^{(t,x)}(\omega))d^0\beta_\tau(\omega),$$

$$T\geqslant s\geqslant t\geqslant 0.$$

The β^+-differentiability of $\xi_s^{(t,x)}(\omega)$ with respect to the time parameter s has already been shown in the previous example. Let us consider the β-differentiability with respect to the time parameter t, which denotes the initial time of the diffusion process. Because the diffusion process $\xi_s^{(t,x)}(\omega)$ is $B_{[s,t]}\times N_s^t$-measurable with respect to (t,ω) for a fixed s, we must concern ourselves with its β^--differentiability in the L_{2n}-sense. For the simplicity of the discussion, we shall put on the coefficients $a(t,x)$ and $b(t,x)$ one more assumption, that is, we shall assume that these functions are continuously differentiable with respect to x and that their derivatives are bounded on $[s,t]\times R^1$.

Then with the help of Theorem 3 (see later), we can conclude that the process is β^--differentiable with respect to t for a fixed $s(t\leqslant s\leqslant T)$ and its β^--derivative is given as the unique solution of the following stochastic integral equation.

$$\zeta_s^t(\omega)+b(t,x)=\int_t^s a_x(\tau,\xi_\tau^{(t,x)}(\omega))\zeta_\tau^t(\omega)d\tau + \int_t^s b_x(\tau,\xi_\tau^{(t,x)}(\omega))\zeta_\tau^t(\omega)d^0\beta_\tau(\omega), \tag{6}$$

where $\zeta_s^t(\omega)=\frac{\partial^-}{\partial^-\beta_t}\xi_s^{(t,x)}(\omega)$, and a_x, b_x denote $\frac{\partial}{\partial x}a(t,x)$, $\frac{\partial}{\partial x}b(t,x)$.

The assumption put on $a(t,x)$ and $b(t,x)$ yield that the above equation has a unique solution.

Example 3. Let $\varphi(t,x)$ be a bounded and continuously differentiable function whose derivative $\frac{\partial}{\partial x}\varphi$ is bounded on $[0,T]\times R^1$, and let $\xi_t(\omega)$ be the diffusion process determined by the equation (3). Then the process $\varphi(t,\xi_t(\omega))$ is β^+-differentiable on $[0,T]$ in the L_{2n}-sense and its derivative is given as follows.

$$\frac{\partial^+}{\partial^+\beta_t}\varphi(t,\xi_t(\omega))=b(t,\xi_t(\omega))\cdot\frac{\partial}{\partial x}\varphi(t,\xi_t(\omega)). \tag{7}$$

Now let us proceed to a further discussion.

Theorem 2. *If a separable, $B_{[0,t]}\times N_t^0(B_{[t,T]}\times N_T^t)$-measurable stochastic process is $\beta^+(\beta^-)$-differentiable in the L_{2n}-sense $(n\geqslant 2)$, then the process $X_t(\omega)$ is continuous at any point $t(\in[0,T])$ with probability one.*

Theorem 3. *If a $B_{[0,t]}\times N_t^0(B_{[t,T]}\times N_T^t)$-measurable stochastic process $\{X_t(\omega),\ t\in[0,T]\}$ is $\beta^+(\beta^-)$-differentiable in the L_{2n}-sense, then the process is also $\beta^+(\beta^-)$-differentiable in the L_{2m}-sense for $m\leqslant n$ and all the $\beta^+(\beta^-)$-derivatives of the process $X_t(\omega)$ in the L_{2m}-sense $(m=1,2,\cdots,n)$ coincide with each other up to the stochastic equivalence.*

Finally we shall show the rules about the β-derivative computation in the next.

Theorem 4. (i) *If the functions $f_t(\omega)$ and $g_t(\omega)$ are $\beta^+(\beta^-)$-differentiable in the L_{2n}-sense, then the following equality holds.*

$$\frac{\partial^\pm}{\partial^\pm\beta_t}\{C_1f_t(\omega)+C_2g_t(\omega)\}=C_1\frac{\partial^\pm}{\partial^\pm\beta_t}f_t(\omega)+C_2\frac{\partial^\pm}{\partial^\pm\beta_t}g_t(\omega) \tag{8}$$

where C_1 and C_2 are constants.

(ii) *If the functions $f_t(\omega)$ and $g_t(\omega)$ are $\beta^+(\beta^-)$-differentiable in the L_{2n}-sense, and if $M|f_t(\omega)|^{2n}$, $M|g_t(\omega)|^{2n}$ are finite for $t\in[0,T]$, then* *the product* $f_t(\omega)g_t(\omega)$ *is* $\beta^+(\beta^-)$*-differentiable in the* L_{2^n-1}*-sense and the following equality holds.*

$$\frac{\partial^{\pm}}{\partial^{\pm}\beta_t} f_t(\omega)\cdot g_t(\omega)=g_t(\omega)\frac{\partial^{\pm}}{\partial^{\pm}\beta_t} f_t(\omega)+f_t(\omega)\frac{\partial^{\pm}}{\partial^{\pm}\beta_t} g_t(\omega). \tag{9}$$

The notion of the β-derivatives of stochastic processes will be interesting in itself. The discussion about this notion developed above may be, of course, far from the exhaustion but it is enough for our present aim.