151. *Summability of Fourier Series*

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1. Introduction and Theorems.

1.1. Let $\sum a_n$ be an infinite series and (s_n) be the sequence of its partial sums. If

$$L(x)=\frac{1}{-\log(1-x)}\sum_{n=1}^{\infty} s_n x^n/n \to s \quad \text{as} \quad x \uparrow 1,$$

then the series $\sum a_n$ is said to be (L) summable to s. We shall consider a more general summability. Let (p_n) be a sequence of nonnegative numbers and suppose that the series $p(x)=\sum_{n=1}^{\infty} p_n x^n$ converges for all x, $0<x<1$ and $p(x)\uparrow\infty$ as $x\uparrow 1$. If

$$P(x)=\frac{1}{p(x)}\sum_{n=1}^{\infty} p_n s_n x^n \to s \quad \text{as} \quad x \uparrow 1,$$

then the series $\sum a_n$ is said to be (P) summable to s.

About (L) summability of Fourier series, M. Nanda ([1], cf. [2] and [3]) proved the

Theorem I. *If*

$$(1) \qquad g(t)=\int_t^{\pi}\varphi(u)u^{-1}du=o(\log 1/t) \quad \textit{as} \quad t\downarrow 0$$

where $\varphi(u)=f(x_0+u)+f(x_0-u)-2s$, *then the Fourier series of* f *is* (L) *summable to* s *at the point* x_0.

We shall generalize this theorem to (P) summability in the following form.

Theorem 1. *Suppose that the sequence* (np_n) *is monotone (non-decreasing or non-increasing) and concave or convex and that*

$$p(x)/(1-x)^2p'(x)\to\infty \quad \textit{as} \quad x\uparrow 1.$$

If

$$(2) \qquad \int_{1-x}^{\pi} G(t)t^{-3}dt=o(p(x)/(1-x)^2p'(x)) \quad \textit{as} \quad x\uparrow 1$$

where $G(t)=\int_0^t |g(u)|du$, *then the Fourier series of* f *is* (P) *summable to* s *at the point* x_0.

The condition (2) is the consequence of

$$(3) \qquad \int_0^x (p(t)/(1-t)^3p'(t))dt \leqq Ap(x)/(1-x)^2p'(x) \quad \text{as} \quad x\uparrow 1$$

and

(4) $$G(t)=\int_0^t |g(u)|\,du=o(p(1-t)/p'(1-t)) \quad \text{as} \quad t\downarrow 0.$$

Further (3) is the consequence of

(5) $$p(t)/(1-t)^a p'(t)\uparrow \quad \text{as} \quad t\uparrow 1, \quad \text{for an } a<2.$$

The function $p(x)=(-\log(1-x))^b$, b being a positive integer, satisfies the condition of Theorem 1 concerning $p(x)$ and also (5). Thus (4) gives

Corollary 1. *If*

$$G(t)=\int_0^t |g(u)|\,du=o(t\log 1/t) \quad as \quad t\downarrow 0,$$

then the Fourier series of f is (P) summable to s at the point x_0 where $p(x)=(-\log(1-x))^b$, b being a positive integer.

This corollary includes Theorem I as a particular case.

1.2. If $L(x)$ is of bounded variation on an interval $(c,1)$, $0<c<1$, then the series $\sum a_n$ is said to be $|L|$ summable. Similarly, if $P(x)$ is of bounded variation on $(c,1)$, then the series is said to be $|P|$ summable.

Following theorems are known ([4], [5])

Theorem II. *If*

(6) $$\frac{1}{t\log(2\pi/t)}\int_t^\pi \frac{\varphi(u)}{2\sin u/2}\,du=\frac{h(t)}{t\log(2\pi/t)}\in L(0,\pi),$$

then the Fourier series of f is $|L|$ summable at the point x_0.

Theorem III. *Suppose that* (i) *the sequence $(n\,p_n)$ is of bounded variation and that* (ii) *there is an a, $0<a<1$, such that $(1-x)^a p(x)\downarrow$ as $x\uparrow 1$. If $h(t)/t\,p(1-t)\in L(0,\pi)$, then the Fourier series of f is P summable at the point x_0.*

We shall prove the following

Theorem 2. *Suppose that* (i) *$(n\,p_n)$ and $(n^2\,p_n)$ are monotone and concave or convex and that* (ii) *$(1-x)^2p''(x)/p(x)\in L(0,\pi)$. If*

(7) $$\int_0^1 H(t)t^{-3}dt\int_{1-t}^1((1-x)^2p''(x)/p(x))dx<\infty$$

where $H(t)=\int_0^t |h(u)|\,du$, then the Fourier series of f is $|P|$ summable at the point x_0

The condition (7) is satisfied when

(8) $$\int_0^t (u^2p''(1-u)/p(1-u))du\leqq t^3p''(1-t)/p(1-t) \quad \text{for all} \quad t>0$$

and

(9) $$H(t)p''(1-t)/p(1-t)\in L(0,1).$$

If $p(x)=-\log(1-x)$, then the condition (8) is satisfied and the condition (9) becomes (6). Hence Theorem II is a particular case of Theorem 2. More generally, if $p(x)=(-\log(1-x))^b$ (b being a positive integer), then (8) is satisfied and (9) reduces also to (6). Thus we get

Corollary 2. *If the condition* (6) *is satisfied, then Fourier series of* f *is* $|P|$ *summable at the point* x_0, *where* $p(x)=(-\log(1-x))^b$, b *being a positive integer.*

This corollary is not contained in Theorem III, since the sequence (np_n) in Corollary 2 is not of bounded variation. Therefore Theorems III and 2 are mutually exclusive.

2. Proof of Theorem 1.

We can suppose that $\int_0^\pi \varphi(u)du=0$ and $p_1=p_2=0$. The sequence $(np_n;\ n\geqq 3)$ is also monotone and concave or convex. Let s_n be the nth partial sum of the Fourier series of f at the point x_0, then

$$\pi(s_n-s)=\int_0^\pi \varphi(t)t^{-1}\sin nt\,dt+o(1),$$

so that

$$\pi\sum_{n=1}^\infty p_n(s_n-s)x^n=\int_0^\pi \varphi(t)t^{-1}\left(\sum_{n=1}^\infty p_n x^n \sin nt\right)dt+o(p(x)) \quad \text{as} \quad x\uparrow 1$$

We shall prove that the integral on the right side is $o(p(x))$ as $x\uparrow 1$. By integration by parts, the integral equals to

$$\lim_{t\to 0}\left(g(t)\sum_{n=1}^\infty p_n x^n \sin nt\right)+\int_0^\pi g(t)\left(\sum_{n=1}^\infty np_n x^n \cos nt\right)dt=U+V,$$

where $U=0$, since $tg(t)\to 0$ as $t\to 0$ and the series $\sum np_n x^n$ converges. Now,

$$\begin{aligned}
&\sum_{n=1}^\infty np_n x^n \cos nt\\
&=\mathfrak{R}\sum_{n=1}^\infty np_n x^n e^{int}\\
&=-\mathfrak{R}\sum_{n=1}^\infty \Delta(np_n)x^{n+1}e^{i(n+1)t}/(1-xe^{it})\\
&=-\sum_{n=1}^\infty \Delta(np_n)x^{n+1}\frac{\cos(n+1)t-x\cos nt}{(1-x)^2+4x\sin^2 t/2}\\
&=\frac{-1}{(1-x)^2+4x\sin^2 t/2}\\
&\quad\times\left(\sum_{n=1}^\infty \Delta(np_n)x^{n+1}\cos(n+1)t-x^2\sum_{n=1}^\infty \Delta((n+1)p_{n+1})x^{n+1}\cos(n+1)t\right)\\
&=\frac{-1}{(1-x)^2+4x\sin^2 t/2}\\
&\quad\times\left(\sum_{n=1}^\infty \Delta^2(np_n)x^{n+1}\cos(n+1)t+(1-x)^2\sum_{n=1}^\infty \Delta((n+1)p_{n+1})x^{n+1}\cos(n+1)t\right)
\end{aligned}$$

and then

$$V\leqq A\left((1-x)^{-2}\int_0^{1-x}|g(t)|\,dt+\int_{1-x}^\pi |g(t)|\,t^{-2}dt\right)$$
$$\cdot\left(\left|\sum_{n=1}^\infty \Delta^2(np_n)x^{n+1}\right|+(1-x)\left|\sum_{n=1}^\infty \Delta(np_n)x^n\right|+A\right).$$

Since

$$\sum \Delta(np_n)x^{n+1} = -(1-x)\sum np_n x^n = -x(1-x)p'(x)$$

and

$$\sum \Delta^2(np_n)x^{n+1} = -(1-x)\sum \Delta(np_n)x^n = (1-x)^2 p'(x),$$

we get

$$V \leqq Ap'(x)\left(\int_0^{1-x} |g(t)|\,dt + (1-x)^2\int_{1-x}^{\pi} |g(t)|\,t^{-2}dt\right)$$
$$\leqq A(1-x)^2 p'(x)\left(\int_{1-x}^{\pi} G(t)t^{-3}dt + G(\pi)/\pi^2\right) = o(p(x))$$

as $x \uparrow 1$, by (2) and $(1-x)^2 p'(x) = o(p(x))$. Thus the theorem is proved.

3. Proof of Theorem 2.

We shall take $s=0$ in the definition of φ and $p_1=p_2=0$, and suppose that $\int_0^{\pi}\varphi(u)du=0$. Then

$$P(x) = \frac{1}{p(x)}\sum_{n=1}^{\infty} p_n s_n x^n = \frac{1}{\pi p(x)}\int_0^{\pi} \frac{\varphi(t)}{2\sin t/2}\left(\sum_{n=1}^{\infty} p_n \sin(n+1/2)tx^n\right)dt.$$

By differentiation with respect to x,

$$P'(x) = \int_0^{\pi}\frac{\varphi(t)}{2\sin t/2}\left(\sum_{n=1}^{\infty} p_n \sin(n+1/2)t(x^n/p(x))'\right)dt$$
$$= \int_0^{\pi} h(t)\left(\sum_{n=1}^{\infty}(n+1/2)p_n\cos(n+1/2)t\ (x^n/p(x))'\right)dt$$

and then

$$\int_0^1 |P'(x)|\,dx \leqq A\int_0^{\pi}|h(t)|\,dt\int_c^1\left|\sum_{n=1}^{\infty} np_n\cos(n+1/2)t(x^n/p(x))'\right|dx$$
$$+A\int_0^{\pi}|h(t)|\,dt\int_c^1\left|\sum_{n=1}^{\infty} p_n\cos(c+1/2)t(x^n/p(x))'\right|dx$$
$$=Q+R.$$

We shall prove that Q and R are finite, which proves the theorem. Now, the infinite sum in Q is

$$s = \frac{1}{p(x)}\sum_{n=1}^{\infty} n^2 p_n\cos(n+1/2)tx^{n-1} - \frac{p'(x)}{(p(x))^2}\sum_{n=1}^{\infty} np_n\cos(n+1/2)tx^n$$
$$=T-U$$

where

$$p(x)T = \mathfrak{R}\sum_{n=1}^{\infty} n^2p_nx^{n-1}e^{i(n+1/2)t} = -\mathfrak{R}\left(e^{3it}\sum_{n=1}^{\infty}\Delta(n^2p_n)x^ne^{int}/(1-xe^{it})\right)$$
$$= -\sum_{n=1}^{\infty}\Delta(n^2p_n)x^n\frac{\cos(n+3/2)t - x\cos(n+1/2)t}{(1-x)^2+4\sin^2 t/2}$$
$$= \frac{-1}{(1-x)^2+4\sin^2 t/2}\left(\sum_{n=1}^{\infty}\Delta^2(n^2p_n)x^n\cos(n+3/2)t\right.$$
$$\left.+(1-x)^2\sum_{n=1}^{\infty}\Delta((n+1)^2p_{n+1})x^n\cos(n+3/2)t\right)$$

and similarly

$$\frac{(p(x))^2}{p'(x)}U = \frac{-1}{(1-x)^2+4\sin^2 t/2}\left(\sum_{n=1}^{\infty}\Delta^2(np_n)x^{n+1}\cos(n+3/2)t\right.$$

$$+(1-x)^2\sum_{n=1}^{\infty}\Delta((n+1)p_{n+1})x^{n+1}\cos(n+3/2)t\Big).$$

Since $p(x)\leqq p'(x)\leqq p''(x)$ and

$$\sum \Delta(n^2p_n)x^n=-(1-x)p''(x)-\frac{(1-x)}{x}p'(x),$$

$$\sum \Delta^2(n^2p_n)x^n=(1-x)^2p''(x)+\frac{(1-x)^2}{x}p'(x),$$

we get

$$|S|\leqq\frac{A(1-x)^2}{(1-x)^2+t^2}\left(\frac{p''(x)}{p(x)}+\left(\frac{p'(x)}{p(x)}\right)^2\right)\leqq\frac{A(1-x)^2}{(1-x)^2+t^2}\frac{p''(x)}{p(x)},$$

on $(c,1)$. Now

$$\begin{aligned}
Q&=A\int_0^{\pi}|h(t)|\,dt\int_c^1|S|\,dx\\
&=A\int_0^{1-t}|h(t)|\,dt\left(\int_c^{1-t}\frac{p''(x)}{p(x)}dx+\frac{1}{t^2}\int_{1-t}^1(1-x)^2\frac{p''(x)}{p(x)}dx\right)\\
&\quad+A\int_{1-c}^{\pi}|h(t)|\,dt\int_c^1(1-x)^2\frac{p''(x)}{p(x)}dx\\
&=A\int_c^1\frac{p''(x)}{p(x)}dx\int_0^{1-x}|h(t)|\,dt+A\int_c^1(1-x)^2\frac{p''(x)}{p(x)}dx\int_{1-x}^{1-c}\frac{|h(t)|}{t^2}dt+A\\
&\leqq A\int_c^1(1-x)^2\frac{p''(x)}{p(x)}dx\int_{1-x}^{1-c}\frac{H(t)}{t^3}dt+A\\
&\leqq A\int_0^{1-c}\frac{H(t)}{t^3}dt\int_{1-t}^1(1-x)^2\frac{p''(x)}{p(x)}dx+A\leqq A
\end{aligned}$$

by (7). Similarly R is also finite and then $P(x)$ is of bounded variation, which is to be proved.

References

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