11. On the Sum $\sum_{n\leq x}\frac{\tilde{n}}{n^2}$

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The purpose of this note is to present an asymptotic formula for the sum described in the title, where we denote by $\tilde{n}$, for every positive integer n, the maximal square-free divisor of n. We shall prove that

$$(1)\qquad \sum_{n\leq x}\frac{\tilde{n}}{n^2}=A\log x+B+O\left(\frac{\log 2x}{\sqrt{x}}\right)\qquad (x>1),$$

where A and B are constants given by

$$A=\prod_p\left(1-\frac{1}{p(p+1)}\right)$$

(the product being taken over all primes p) and

$$B=\frac{\pi^2}{6}c_3-\frac{6}{\pi^2}A\sum_{m=1}^{\infty}\frac{\log m}{m^2}$$

with the constant c_3 determined by (7), (5) and (3).

We note that the asymptotic formula (1) will immediately give a solution to a problem posed by D. Suryanarayana.*)

1. In this paragraph, t denotes an arbitrary real number >1 and k a fixed square-free integer >0. As usual, we denote by $\varphi(k)$ the Euler totient function of k, by $\sigma(k)$ the sum of all positive divisors of k, and by $v(k)$ the number of distinct prime divisors of k. Also, O-constants are all absolute.

It is well known that

$$\sum_{m\leq t}\frac{1}{m}=\log t+C+O\left(\frac{1}{t}\right),$$

where C is the Euler constant. Using this and the well-known property of the Möbius function $\mu(n)$, namely, $\sum_{d\mid n}\mu(d)=1$ for $n=1$ and $=0$ for $n>1$, we find easily

$$(2)\qquad \sum_{\substack{m\leq t\\(m,k)=1}}\frac{1}{m}=\frac{\varphi(k)}{k}\log t+c_1(k)+O\left(\frac{2^{v(k)}}{t}\right)$$

with

$$(3)\qquad c_1(k)=C\frac{\varphi(k)}{k}-\sum_{d\mid k}\frac{\mu(d)\log d}{d}=O\,(\text{loglog}\,3k).$$

Next, it will follow at once from (2) that

*) Cf. Bull. Amer. Math. Soc., **76**, 976–977 (1970): Problem 17 (2).

(4) $$\sum_{\substack{m\leq t\\(m,k)=1}}\frac{\mu^2(m)}{m}=\frac{6}{\pi^2}\frac{k}{\sigma(k)}\log t+c_2(k)+O\left(\frac{2^{\nu(k)}+\log t}{\sqrt{t}}\right),$$

where

(5) $$c_2(k)=c_1(k)\prod_{p\nmid k}\left(1-\frac{1}{p^2}\right)-2\frac{\varphi(k)}{k}\sum_{\substack{m=1\\(m,k)=1}}^{\infty}\frac{\mu(m)\log m}{m^2}$$
$$=O\,(\log\log 3k).$$

Finally, (4) can be used to obtain

(6) $$\sum_{m\leq t}\frac{\mu^2(m)\varphi(m)}{m^2}=\frac{6}{\pi^2}A\log t+c_3+O\left(\frac{\log 2t}{\sqrt{t}}\right)$$

with

(7) $$c_3=\sum_{m=1}^{\infty}\frac{\mu(m)c_2(m)}{m^2}-\frac{6}{\pi^2}\sum_{m=1}^{\infty}\frac{\mu(m)\log m}{m\sigma(m)},$$

on noticing that

$$\sum_{m\leq t}\frac{\mu^2(m)\varphi(m)}{m^2}=\sum_{m\leq t}\frac{\mu^2(m)}{m}\sum_{d\mid m}\frac{\mu(d)}{d}$$
$$=\sum_{d\leq t}\frac{\mu(d)}{d^2}\sum_{\substack{m\leq x/d\\(m,d)=1}}\frac{\mu^2(m)}{m}.$$

2. We are now able to prove (1). Since

$$\tilde{n}=\sum_{d\mid\tilde{n}}\varphi(d)=\sum_{d\mid n}\mu^2(d)\varphi(d),$$

we have

$$\sum_{n\leq x}\frac{\tilde{n}}{n^2}=\sum_{n\leq x}\frac{1}{n^2}\sum_{d\mid n}\mu^2(d)\varphi(d)$$
$$=\sum_{d\leq x}\frac{\mu^2(d)\varphi(d)}{d^2}\sum_{m\leq x/d}\frac{1}{m^2}$$
$$=\sum_{d\leq x}\frac{\mu^2(d)\varphi(d)}{d^2}\left(\frac{\pi^2}{6}-\sum_{m>x/d}\frac{1}{m^2}\right),$$

where

$$\sum_{d\leq x}\frac{\mu^2(d)\varphi(d)}{d^2}\sum_{m>x/d}\frac{1}{m^2}=\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{x/m<d\leq x}\frac{\mu^2(d)\varphi(d)}{d^2}$$
$$=\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\frac{6}{\pi^2}A\log m+O\left(\frac{\log 2x}{\sqrt{x/m}}\right)\right)$$
$$=\frac{6}{\pi^2}A\sum_{m=1}^{\infty}\frac{\log m}{m^2}+O\left(\frac{\log 2x}{\sqrt{x}}\right)$$

by (6). Hence, using (6) again, we find that

$$\sum_{n\leq x}\frac{\tilde{n}}{n^2}=\frac{\pi^2}{6}\left(\frac{6}{\pi^2}A\log x+c_3\right)-\frac{6}{\pi^2}A\sum_{m=1}^{\infty}\frac{\log m}{m^2}+O\left(\frac{\log 2x}{\sqrt{x}}\right),$$

which is identical with (1).

This completes the proof of (1).

3. A concise presentation for the constant B in (1) can be found in the following way. We define the function $F(s)$ $(s=\sigma+it)$ by

$$F(s)=\prod_p \left(1-\frac{p-1}{p^{s+1}(p^{s+2}-1)}\right),$$

which is analytic for $\sigma>-1/2$. Then $A=F(0)$ and

$$\sum_{n=1}^{\infty}\frac{\tilde{n}}{n^2}\frac{1}{n^s}=F(s)\zeta(s+1) \qquad (\sigma>0),$$

whence follows that

$$B=CF(0)+F'(0).$$

Proof is by an elementary Abelian theorem for Dirichlet series.