80. *Stability, Attraction Properties and Asymptotic Equivalence of Dynamical Systems*

By Shigeo KONO

Department of Mathematics, Josai University

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1. Introduction. The concepts of stabilities and attraction properties such as the attractors and the region of attraction are rather important to determine the behaviors of the abstract dynamical systems defined on a metric space [1], [2].

In this paper we investigate the problem that to what extent the stability properties and the attraction properties are preserved through the asymptotic equivalence.

Main results obtained are Theorems 3.5, 3.6, 3.7, 3.8 and 3.10.

2. Standing notations. R is the real line. R^+ is the set of non-negative real numbers.

X is a metric space with its metric d.

π_α: $X\times R\to X$ is a dynamical system defined on X.

$\pi_\alpha(p, \cdot)$ is the motion through the point p.

$L^+(p, \alpha)=\{x\,;\, x \text{ is a positive limit point of } \pi_\alpha(p, \cdot)\}$.

$J^+(p, \alpha)=\{y \in X\,;\, \exists\{x_n\}\subset X,\ \{t_n\}\subset R, \text{ such that } x_n\to x,\ t_n\to +\infty \text{ and } \pi_\alpha(x_n, t_n)\to y\}$.

$L^+(p, \alpha)$ and $J^+(p, \alpha)$ are called respectively the positive limit set and the first positive prolongational limit set of a motion $\pi_\alpha(p, \cdot)$.

3. The preservation of stability and attraction properties through the asymptotic equivalence.

The asymptotic equivalence between two differential equations has been an interesting subject for many mathematicians [3]. The following Definition 3.1 is a generalization of this concept for the case of abstract dynamical systems.

Definition 3.1. *We say a dynamical system π_α is asymptotically equivalent to π_β on a subset S of X if*

$$(\forall p \in S, \forall q \in S)\quad d(\pi_\alpha(p, t), \pi_\beta(q, t))\to 0 \quad (t\to +\infty)$$

is valid, and denote this fact as follows:

$$\pi_\alpha \sim \pi_\beta \qquad (S).$$

The following proposition is trivial:

Proposition 3.2. *The asymptotic equivalence is symmetric as well as transitive.*

Corollary 3.2.1. *The asymptotic equivalence of the dynamical systems on a singleton is an equivalence relation.*

Proposition 3.3. *X is a locally compact metric space. S is a non-empty subset of X such that $L^+(p,\alpha)$ and $L^+(p,\beta)$ are both non-empty and compact for any $p \in S$.*

Then,

$$\pi_\alpha \sim \pi_\beta \quad (S) \Longleftrightarrow (\forall p, q \in S) \quad L^+(p,\alpha) \cap L^+(q,\beta) \neq \phi.$$

Proof ($\Rightarrow$). The assumption implies that

$$(\forall p, q \in S) \quad d(\pi_\alpha(p,t), \pi_\beta(q,t)) \to 0 \quad (t \to +\infty).$$

$L^+(p,\alpha)$ and $L^+(q,\beta)$ are both non-empty and compact in X which is a locally compact metric space, so that

$$d(L^+(p,\alpha), \pi_\alpha(p,t)) \to 0 \qquad (t \to +\infty) \tag{1}$$

and

$$d(\pi_\beta(q,t), L^+(q,\beta)) \to 0 \qquad (t \to +\infty) \tag{2}$$

hold (Refer to 1.3.17 of [4]).

Applying (1) and (2) to the inequality

$$d(L^+(p,\alpha), L^+(q,\beta)) \leqq d(L^+(p,\alpha), \pi_\alpha(p,t)) + d(\pi_\alpha(p,t), \pi_\beta(q,t)) + d(\pi_\beta(q,t), L^+(q,\beta)),$$

we know that

$$(\forall p, q \in S) \quad d(L^+(p,\alpha), L^+(q,\beta)) = 0, \tag{3}$$

which implies that

$$(\forall p, q \in S) \quad L^+(p,\alpha) \cap L^+(q,\beta) \neq \phi. \tag{4}$$

($\Leftarrow$) The assumption (4) implies (3). Applying (1), (2) and (3) to the inequality

$$d(\pi_\alpha(p,t), \pi_\beta(q,t)) \leqq d(\pi_\alpha(p,t), L^+(p,\alpha)) + d(L^+(p,\alpha), L^+(q,\beta)) + d(L^+(q,\beta), \pi_\beta(q,t))$$

we know that

$$(\forall p, q \in S) \quad d(\pi_\alpha(p,t), \pi_\beta(q,t)) \to 0 \quad (t \to +\infty),$$

i.e., $\pi_\alpha \sim \pi_\beta \quad (S)$. Q.E.D.

Corollary 3.3.1. *X is a compact metric space. S is a non-empty subset of X.*

Then,

$$\pi_\alpha \sim \pi_\beta \quad (S) \Longleftrightarrow (\forall p, q \in S) \quad L^+(p,\alpha) \cap L^+(q,\beta) \neq \phi.$$

Definition 3.4 [4]. A compact subset M of X is called as follows:

a weak attractor of the dynamical system π_α, if

$$\exists \varepsilon > 0; \quad (\forall x \in K(M,\varepsilon)) \quad L^+(x,\alpha) \cap M \neq \phi,$$

where $K(M,\varepsilon) = \{x; d(x,M) < \varepsilon\}$;

an attractor of π_α, if

$$\exists \varepsilon > 0; \quad (\forall x \in K(M,\varepsilon)) \quad \phi \neq L^+(x,\alpha), \quad L^+(x,\alpha) \subset M;$$

a weakly uniform attractor of π_α, if

$$\exists \varepsilon > 0; \quad (\forall x \in K(M,\varepsilon)) \quad J^+(x,\alpha) \cap M \neq \phi;$$

a uniform attractor of π_α, if

$$\exists \varepsilon > 0; \quad (\forall x \in K(M,\varepsilon)) \quad J^+(x,\alpha) \neq \phi, \quad J^+(x,\alpha) \subset M;$$

stable with respect to π_α, if

$$(\forall \varepsilon>0) \quad \exists \delta>0; \quad \pi_\alpha(K(M,\delta), R^+)\subset K(M,\varepsilon);$$

eventually stable with respect to π_α, *if*

$$(\forall \varepsilon>0) \quad \exists T>0, \delta>0; \quad \pi_\alpha(K(M,\delta), [T, +\infty))\subset K(M,\varepsilon);$$

asymptotically stable with respect to the dynamical system π_α, *if* M *is an attractor of* π_α *as well as stable with respect to* π_α;

weakly asymptotically stable with respect to a dynamical system π_α, *if* M *is a weak attractor of* π_α *as well as eventually stable with respect to* π_α.

In the following, we discuss the problems that to what extent these properties given in Definition 3.4 are preserved through the asymptotic equivalence.

Theorem 3.5. 1) *X is a compact metric space.*

2) *π_α is asymptotically equivalent to π_β on a non-empty open subset S of X.*

3) *M is an attractor of π_α.*

4) *$M\subset S$.*

Then, M is a weak attractor of π_β.

Proof. By the assumption 3), there exists a $\varepsilon>0$ such that

$$(\forall x\in K(M,\varepsilon)) \quad L^+(x,\alpha)\neq\phi, \quad L^+(x,\alpha)\subset M.$$

On the other hand for every point $x\in M$ there exists a neighborhood $U(x)$ such that $U(x)\subset S$. We take a Lebesgue number λ of the covering $\{U(x); x\in M, U(x)\subset S\}$. Then

$$K(M,\lambda)=\bigcup_{x\in M} K(x,\lambda)\subset \bigcup_{x\in M} U(x)\subset S.$$

We can choose ε to satisfy $\varepsilon\leqq\lambda$.

Thus
$$K(M,\varepsilon)\subset S. \tag{1}$$

Then

$$(\forall p\in K(M,\varepsilon), \forall q\in S) \qquad L^+(p,\alpha)\cap L^+(q,\beta)\neq\phi,$$

because of (1), the assumption 2) and Corollary 3.3.1. On the other hand

$$(\forall p\in K(M,\varepsilon)) \quad L^+(p,\alpha)\neq\phi, \quad L^+(p,\alpha)\subset M.$$

$$\therefore \quad (\forall q\in K(M,\varepsilon)) \qquad L^+(q,\beta)\cap M\neq\phi.$$

This shows that M is a weak attractor of π_β. Q.E.D.

Theorem 3.6.

1) *X is a locally compact metric space.*

2) *M is a uniform attractor of π_α.*

3) *S is an open subset of X such that $M\subset S$.*

4) *$J^+(x,\beta)$ is non-empty and compact for any $x\in S$.*

5) $\pi_\alpha\sim\pi_\beta \quad (S)$.

Then M is a weakly uniform attractor of π_β.

Proof. The assumption 2) implies that

$$\exists\varepsilon>0; \quad x\in K(M,\varepsilon)\Rightarrow J^+(x,\alpha)\neq\phi, \quad J^+(x,\alpha)\subset M.$$

Here we choose the ε to satisfy the condition $K(M,\varepsilon)\subset S$. Then for any $x \in K(M, \varepsilon)$ $J^+(x, \alpha)$ is compact and non-empty in X which is a locally compact metric space, so that $L^+(x, \alpha)$ is non-empty and compact. (Refer to 2.3.14 of [4].)

On the other hand

$$(\forall x, y \in K(M, \varepsilon)) \qquad L^+(x, \alpha) \cap L^+(y, \beta) \neq \phi, \tag{1}$$

because of Proposition 3.3, the assumption 4) and 5). Applying

$$L^+(y, \beta) \subset J^+(y, \beta)$$

and

$$(\forall x \in K(M, \alpha)) \qquad L^+(x, \alpha) \subset J^+(x, \alpha) \subset M$$

to (1), we can find that

$$(\forall y \in K(M, \varepsilon)) \qquad J^+(y, \beta) \cap M \neq \phi.$$

Thus M is a weakly uniform attractor of π_β. Q.E.D.

Theorem 3.7. 1) *X is a compact metric space.*

2) *S is a non-empty open subset of X.*

3) *M is a compact subset of X and stable with respect to π_α.*

4) *$M \subset S$.*

5) *$\pi_\alpha \sim \pi_\beta \quad (S)$.*

Then, M is eventually stable with respect to π_β.

Proof. We can find a $\delta' > 0$ such that

$$K(M, \delta') \subset S. \tag{1}$$

Because of the stability of M with respect to π_α, the following condition holds:

$$(\forall \varepsilon > 0) \quad \exists \delta'' > 0; \quad \pi_\alpha(K(M, \delta''), R^+) \subset K\left(M, \frac{\varepsilon}{2}\right).$$

Thus, taking δ to be $\min\{\delta', \delta''\}$,

$$(\forall \varepsilon > 0) \quad \exists \delta > 0; \begin{cases} 1) & \pi_\alpha(K(M, \delta), R^+) \subset K\left(M, \dfrac{\varepsilon}{2}\right), \\ 2) & K(M, \delta) \subset S. \end{cases}$$

On the other hand, the assumption 5) implies that for any $q \in S$ and for any $p \in K(M, \delta)$ the condition

$$(\forall \varepsilon > 0) \quad \exists T > 0; \quad t \geqq T \Rightarrow d(\pi_\alpha(p, t), \pi_\beta(q, t)) < \frac{\varepsilon}{2}$$

is valid.

Applying these results to the inequality

$$d(\pi_\beta(q, t), M) \leqq d(\pi_\beta(q, t), \pi_\alpha(p, t)) + d(\pi_\alpha(p, t), M),$$

we can find the fact that

$$(\forall \varepsilon > 0) \quad \exists\, (T > 0, \delta > 0); \quad (\forall q \in K(M, \delta), \forall t \in [T, +\infty))$$

$$d(\pi_\beta(q, t), M) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that M is eventually stable with respect to π_β.

Q.E.D.

Theorem 3.8. 1) *X is a compact metric space,*

2) *S is a non-empty open subset of X*,

3) $\pi_\alpha \sim \pi_\beta \quad (S)$,

4) *M is a compact subset of X and asymptotically stable with respect to* π_α,

5) $M \subset S$.

Then, M is weakly asymptotically stable respect to π_β.

We omit the proof, which is, however, easy.

Definition 3.9 [2]. Let M be a compact subset of X.

$$A_w(M, \alpha) = \{x \in X\,;\, L^+(x, \alpha) \cap M \neq \phi\}.$$
$$A(M, \alpha) = \{x \in X\,;\, L^+(x, \alpha) \neq \phi, L^+(x, \alpha) \subset M\}.$$
$$A_u(M, \alpha) = \{x \in X\,;\, L^+(x, \alpha) \neq \phi, J^+(x, \alpha) \subset M\}.$$

we call $A_w(M, \alpha)$, $A(M, \alpha)$ and $A_u(M, \alpha)$ the region of weak attraction, the region of attraction and the region of uniform attraction, of M respectively.

Theorem 3.10. 1) *X is a locally compact metric space.*

2) $\phi \neq S \subset X$.

3) $\pi_\alpha \sim \pi_\beta \quad (S)$.

4) *M is a non-empty compact subset of X.*

Then,

1) $A_w(M, \alpha) \cap S \neq \phi \Rightarrow S \subset A_w(M, \beta)$

2) $A(M, \alpha) \cap S \neq \phi \Rightarrow S \subset A(M, \beta)$

3) $A_u(M, \alpha) \cap S \neq \phi \Rightarrow S \subset A_u(M, \beta)$.

Proof. 1) Let x be a point of $A_w(M, \alpha) \cap S$. Then there exists a sequence $\{t_n\}$ such that

(1) $$t_n \to +\infty \qquad (n \to +\infty),$$

(2) $$d(\pi_\alpha(x, t_n), M) \to 0 \qquad (n \to +\infty) \quad [5],$$

(3) $$(\forall y \in S) \quad d(\pi_\alpha(x, t_n), \pi_\beta(y, t_n)) \to 0 \quad (n \to +\infty).$$

On the other hand

$$(\forall y \in S) \quad d(\pi_\beta(y, t_n), M) \leqq d(\pi_\beta(y, t_n), \pi_\alpha(x, t_n)) + d(\pi_\alpha(x, t_n), M).$$

Thus

$$(\forall y \in S) \quad \exists \{t_n\}\,;\, t_n \to +\infty \quad (n \to +\infty), \quad d(\pi_\beta(y, t_n), M) \to 0,$$

which implies that $S \subset A_w(M, \beta)$.

2) Let x be a point of $A(M, \alpha) \cap S$. Then,

$$d(\pi_\alpha(x, t), M) \to 0 \qquad (t \to +\infty) \quad [5]$$

and

$$(\forall y \in S) \quad d(\pi_\alpha(x, t), \pi_\beta(y, t)) \to 0 \quad (t \to +\infty)$$

are valid. Applying these results to the inequality

$$d(\pi_\beta(y, t), M) \leqq d(\pi_\beta(y, t), \pi_\alpha(x, t)) + d(\pi_\alpha(x, t), M),$$

we can find that

$$(\forall y \in S) \quad d(\pi_\beta(y, t), M) \to 0 \quad (t \to +\infty).$$

Thus

$$S \subset A(M, \beta).$$

3) Let x be a point of $A_u(M, \alpha) \cap S$. For any neighborhood $V(M)$

of M there exists a neighborhood $U(x)$ of x and a positive number T such that

$$(\forall t \geqq T) \qquad \pi_\alpha(U(x), t) \subset V(M) \quad [5].$$

By the assumption 3) the following fact is valid:

$$(\forall y \in S) \quad (\forall \varepsilon > 0) \quad \exists T' \geqq T;\ t \geqq T' \Rightarrow d(\pi_\beta(y, t), \pi_\alpha(x, t)) < \varepsilon.$$

Here

$$\pi_\alpha(x, t) \in \pi_\alpha(U(x), t) \subset V(M),$$

because $t \geqq T$.

As $V(M)$ is open, for sufficiently small $\varepsilon > 0$ there exists a T' such that

$$(\forall t \geqq T') \qquad \pi_\beta(y, t) \in V(M),$$

which implies that

$$(\forall t \geqq T') \qquad y \in \pi_\beta(V(M), -t).$$

$\pi_\beta(\cdot, -t)$ is a homeomorphism on X, so that $\pi_\beta(V(M), -t)$ is open, and so we have a neighborhood $U(y)$ of y such that

$$U(y) \subset \pi_\beta(V(M), -t),$$

which implies that

$$\pi_\beta(U(y), t) \subset V(M).$$

Thus we can conclude that

$$(\forall y \in S) \quad (\forall V(M)) \quad \exists (U(y), T' > 0); \quad (\forall t \geqq T') \quad \pi_\beta(U(y), t) \subset V(M).$$

Therefore $S \subset A_u(M, \beta)$. Q.E.D.

Corollary 3.10.1. *Under the same assumptions as Theorem* 3.10, *the followings are valid*:

1) $S \cap A_w(M, \alpha) \neq \phi \Rightarrow S \subset A_w(M, \alpha)$,
2) $S \cap A(M, \alpha) \neq \phi \Rightarrow S \subset A(M, \alpha)$,
3) $S \cap A_u(M, \alpha) \neq \phi \Rightarrow S \subset A_u(M, \alpha)$.

The proof is easy, using Proposition 3.2 and Theorem 3.10.

Corollary 3.10.2. *Under the same assumptions as Theorem* 3.10, *the followings are valid*:

1) $A_w(M, \alpha) \subset S \Rightarrow A_w(M, \alpha) \subset A_w(M, \beta)$,
2) $A(M, \alpha) \subset S \Rightarrow A(M, \alpha) \subset A(M, \beta)$,
3) $A_u(M, \alpha) \subset S \Rightarrow A_u(M, \alpha) \subset A_u(M, \beta)$.

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