110. *An Analogue of the Paley-Wiener Theorem for the Heisenberg Group*

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1. Introduction. Let $\boldsymbol{R}$ (resp $\boldsymbol{C}$) be the real (resp. complex) number field as usual. Let G be the n-th Heisenberg group, i.e. the group of all real matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1}$$

where $a=(a_1,\cdots,a_n)\in \boldsymbol{R}^n$, $b={}^t(b_1,\cdots,b_n)\in \boldsymbol{R}^n$, $c\in \boldsymbol{R}$ and I_n is the identity matrix of n-th order. Let H be the abelian normal subgroup consisting of the elements of the form (1.1) with $a=0$. For any real η we denote by χ_η the unitary character of H defined by $\chi_\eta:\begin{pmatrix} 1 & 0 & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix}\to e^{2\pi i\eta c}$. Let U^η be the unitary representation of G induced by χ_η. Then the Plancherel theorem can be proved by means of $U^\eta(\eta\in \boldsymbol{R})$ (see e.g. [4]). However, as we have seen in the case of euclidean motion group ([2]), in order to prove an analogue of the Paley-Wiener theorem we have to consider the representations which have more parameters.

Let $\hat{H}$ be the dual group of H. In this paper we consider the Fourier transform defined on $\hat{H}\cong \boldsymbol{R}^{n+1}$.

Let $C_c^\infty(G)$ be the set of all infinitely differentiable functions on G with compact support. For any $\xi\in \boldsymbol{R}^n$ and $\eta\in \boldsymbol{R}$ we denote by $U^{\xi,\eta}$ the unitary representation of G induced by the unitary character $\chi_{\xi,\eta}$ of H: $\chi_{\xi,\eta}\begin{pmatrix} 1 & 0 & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix}=e^{2\pi i\langle \xi,b\rangle+2\pi i\eta c}$. We define the (operator valued) Fourier transform T_f of $f\in C_c^\infty(G)$ by

$$T_f(\xi,\eta)=\int_G f(g)U_g^{\xi,\eta}dg,$$

where dg is the Haar measure on G. Then $T_f(\xi,\eta)$ is an integral operator on $L_2(\boldsymbol{R}^n)$ (§2). Denote by $K_f(\xi,\eta;x,y)$ $(x,y\in \boldsymbol{R}^n)$ be the kernel function of $T_f(\xi,\eta)$. We shall call K_f the scalar Fourier transform of f.

The purpose of this paper is to determine the image of $C_c^\infty(G)$ by the scalar Fourier transform (analogue of the Paley-Wiener theorem).

I. M. Gel'fand has investigated the scalar Fourier transform on the Lorentz group and proved the Paley-Wiener theorem for the class of rapidly decreasing functions [1].

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2. The scalar Fourier transform. Let $L_2(\boldsymbol{R}^n)$ be the Hilbert space of all square integrable functions on $\boldsymbol{R}^n$. Let $\langle\,,\rangle$ be the inner product of the n-dimensional euclidean space $\boldsymbol{R}^n$. Let us realize the unitary representation $U^{\xi,\eta}(\xi\in\boldsymbol{R}^n,\eta\in\boldsymbol{R})$ on $L_2(\boldsymbol{R}^n)$. For an element $g\in G$ of the form (1.1), we define $U_g^{\xi,\eta}$ by the formula

$$(U_g^{\xi,\eta}F)(x)=e^{2\pi i\langle\xi,b\rangle+2\pi i\eta(c-\langle x,b\rangle)}F(x-a),$$

$(F\in L_2(\boldsymbol{R}^n), x\in\boldsymbol{R}^n)$. Then $U^{\xi,\eta}$ is a unitary representation of G.

Lemma 1. *If $\eta\neq 0, U^{0,\eta}$ is an irreducible unitary representation of G.*

For the proof of this lemma, see e.g. [4].

Let R_z be the right translation of $L_2(\boldsymbol{R}^n)$ by $z\in\boldsymbol{R}^n$: $(R_zF)(x)=F(x+z)$. Then it can be shown that if $\eta\neq 0$, $R_{(1/\eta)(\xi-\xi')}U_g^{\xi,\eta}=U^{\xi',\eta}R_{(1/\eta)(\xi-\xi')}$ for every $g\in G$ and for every $\zeta,\xi'\in\boldsymbol{R}^n$. Thus by Lemma 1 we have the following

Lemma 2. *If $\eta\neq 0, U^{\xi,\eta}$ is irreducible and $U^{\xi,\eta}$ is equivalent to $U^{\xi',\eta}$ by $R_{(1/\eta)(\xi-\xi')}$ for any $\xi,\xi'\in\boldsymbol{R}^n$.*

We normalize the Haar measure dg on G such that

$$dg=da_1\cdots da_n db_1\cdots db_n dc \qquad \text{for } g=\begin{pmatrix}1 & a & c\\ 0 & I_n & b\\ 0 & 0 & 1\end{pmatrix}.$$

Then we have

$$T_f(\xi,\eta)F(x)=\int_{\boldsymbol{R}^n}K_f(\xi,\eta\,;x,y)F(y)dy,\qquad (F\in L_2(\boldsymbol{R}^n))$$

where $dy=dy_1\cdots dy_n$ and

$$K_f(\xi,\eta\,;x,y)=\int_{\boldsymbol{R}^{n+1}}f\begin{pmatrix}1 & x-y & c+\langle x,b\rangle\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}e^{2\pi i(\langle\xi,b\rangle+\eta c)}dbdc \tag{2.1}$$

Let $\mathfrak{F}$ be the set of all infinitely differentiable functions $\Phi(x,y)$ on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ such that $\Phi_0(y)$, which we define by $\Phi_0(y)=\Phi(0,y)$, are functions of $y\in\boldsymbol{R}^n$ with compact support. For any $r\geqq 0$, put $\mathfrak{F}_r=\{\Phi\in\mathfrak{F}\,;\ \mathrm{supp}\,(\Phi_0)\subset\{y\in\boldsymbol{R}^n\,;|y_j|\leqq r, j=1,\cdots,n\}\}$. And for any $r\geqq 0$ we denote by B_r the set of all elements $g=\begin{pmatrix}1 & a & c\\ 0 & I_n & b\\ 0 & 0 & 1\end{pmatrix}\in G$ such that $|a_j|\leqq r, |b_j|\leqq r$ $(j=1,\cdots,n)$ and $|c|\leqq r$. Then we have the following

Lemma 3. *For any $\xi\in\boldsymbol{R}^n$ and $\eta\in\boldsymbol{R}, K_f(\xi,\eta\,;x,y)\in\mathfrak{F}_r$ as a function of $x,y\in\boldsymbol{R}^n$, whenever $f\in C_c^\infty(G)$ and* $\mathrm{supp}\,(f)\subset B_r$.

By this lemma we can define a $\mathfrak{F}$-valued function K_f on $\hat{H}\cong\boldsymbol{R}^{n+1}$ by

$$K_f[\xi, \eta](x, y) = K_f(\xi, \eta\,; x, y).$$

We shall call K_f *the scalar Fourier transform* of f.

For any $z \in \boldsymbol{R}^n$, we define an operator L_z on $\mathfrak{F}$ by $(L_z\Phi)(x, y) = \Phi(x-y, y-z)$.

Lemma 4. *Suppose that $f \in C_c^\infty(G)$. Then we have*

(i) *if $\eta \neq 0, K_f[\xi, \eta] = L_{(1/\eta)(\xi-\xi')} K_f[\xi', \eta]$ for every $\xi, \xi' \in \boldsymbol{R}^n$,*

(ii) *$K_f[\xi, 0] = L_z K_f[\xi, 0]$ for every $z, \xi \in \boldsymbol{R}^n$.*

From Lemma 2 we can prove (i). The statement (ii) is an immediate consequence of (2.1).

3. The analogue of the Paley-Wiener theorem. Let K be a $\mathfrak{F}$-valued function on $\hat{H}^c \cong \boldsymbol{C}^{n+1}$. We shall call that K is entire holomorphic if $K[\zeta, \omega](x, y)$ is an entire holomorphic function of $(\zeta, \omega) \in \boldsymbol{C}^{n+1}$ for every $x, y \in \boldsymbol{R}^n$. For any polynomial $q(y_1, \cdots, y_n)$ on $\boldsymbol{R}^n$ we denote $q(D_y) = q(\partial/\partial y_1, \cdots, \partial/\partial y_n)$.

Theorem. *A $\mathfrak{F}$-valued function K on $\hat{H}^n (\cong \boldsymbol{R}^{n+1})$ is the scalar Fourier transform of $f \in C_c^\infty(G)$ such that* supp$(f) \subset B_r$ *if and only if it satisfies the following conditions*:

(I) *$K[\xi, \eta] \in \mathfrak{F}_r$ for any $\xi \in \boldsymbol{R}^n, \eta \in \boldsymbol{R}$;*

(II) (i) *If $\eta \neq 0, K[\xi, \eta] = L_{(1/\eta)\xi} K[0, \eta]$ for any $\xi \in \boldsymbol{R}^n$,*

(ii) *$K[\xi, 0] = L_z K[\xi, 0]$ for any $\xi, z \in \boldsymbol{R}^n$;*

(III) *K can be extended to an entire holomorphic function on $\hat{H}^c$;*

(IV) *For any polynomial function p on $\hat{H}^c$ and for any polynomial q on $\boldsymbol{R}^n$, there exists a constant $C_{p,q}$ such that*

$$|p(\zeta, \omega) q(D_y) K[\zeta, \omega](0, y)| \leqq C_{p,q} \exp 2\pi r \left(\sum_{j=1}^n |\mathrm{Im}\, \zeta_j| + |\mathrm{Im}\, \omega| \right)$$

for every $\zeta \in \boldsymbol{C}^n$ and $\omega \in \boldsymbol{C}$.

The necessity of the theorem follows from the facts mentioned in §2.

Let us assume that K is an arbitrary $\mathfrak{F}$-valued function on $\hat{H}$ satisfying the conditions (I)–(IV) in the theorem. Define a function f on G by

$$f\begin{pmatrix} 1 & a & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} = \int_{\boldsymbol{R}^{n+1}} K[\xi, \eta](0, -a) e^{-2\pi i\langle \xi, b\rangle - 2\pi i \eta c} d\xi d\eta,$$

where $d\xi = d\xi_1 \cdots d\xi_n$.

Making use of the condition (I), and the classical Paley-Wiener theorem ([3]), it can be shown that supp$(f) \subset B_r$.

The differentiability of f follows from (IV) and the Lebesgue's theorem.

Finally we have to check that $K_f = K$ which can be shown using the functional equations (II).

References

[1] I. M. Gel'fand: On the structure of the ring of rapidly decreasing functions on a Lie group (in Russian). Doklady Akad. Nauk S. S. S. R., **124**, 19 (1959).

[2] K. Kumahara and K. Okamoto: An analogue of the Paley-Wiener theorem for the euclidean motion group (to appear).

[3] R. Paley and N. Wiener: Fourier Transforms in the Complex Domain. Amer. Math. Soc. Colloquium Publ. New York (1934).

[4] L. Pukanszky: Leçons sur les représentations des groupes. Dunod, Paris (1967).