21. On the Topological Spaces with the $\mathfrak{B}$-property

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Recently, P. Zenor [9] defined the topological class contained in the countably paracompact spaces. It is the generalization of C. H. Dowker ([1], Theorem 2) or F. Isikawa [2]. On the other hand, S. Sasada [7] defined the α_i-spaces $(i=1,2)$ in addition the normality (normal $\mathfrak{B}$-spaces are α_2-spaces).

The purpose of this paper is to study some characterizations and properties of $\mathfrak{B}$-spaces. F. Isikawa [2] proved the following theorem:

Theorem 1. *In order that a topological space be countably paracompact, it is necessary and sufficient that if a decreasing sequence* $\{F_i \mid i=1,2,\cdots\}$ *of closed sets with vacuous intersection is given, then there exists a decreasing sequence* $\{G_i \mid i=1,2,\cdots\}$ *of open sets such that* $\{\overline{G_i} \mid i=1,2,\cdots\}$ *has a vacuous intersection and* $G_i \supset F_i$ *for* $i=1,2,\cdots$.

At this time, we can naturally define the $\mathfrak{B}$-space, that is, a topological space X is said to be a $\mathfrak{B}$-*space* if every monotone decreasing[1] family $\{F_\alpha \mid \alpha \in A\}$ of closed sets with the vacuous intersection has the monotone decreasing family $\{G_\alpha \mid \alpha \in A\}$ of open sets such that $\bigcap_{\alpha \in A} \overline{G_\alpha} = \emptyset$ and $G_\alpha \supset F_\alpha$ for each $\alpha \in A$. From the above definition, the $\mathfrak{B}$-property is weakly hereditary[2] and the following is trivial:

Proposition. *In order that a topological space X be a $\mathfrak{B}$-space, it is necessary and sufficient that every monotone increasing*[1] *open covering* $\{G_\alpha \mid \alpha < \lambda\}$ *of X has the monotone increasing open covering* $\{U_\alpha \mid \alpha < \lambda\}$ *of X such that* $G_\alpha \supset \overline{U_\alpha}$ *for each* $\alpha < \lambda$.

In order to prove some theorems, we shall use the following:

Lemma. *Let X be a topological space, then X is countably paracompact if and only if every monotone increasing countable open covering* $\mathfrak{U}$ *of X has the σ-cushioned*[3] *open refinement.*

The proof of this lemma is easily seen from Theorem 1.

Theorem 2. *In a topological space X, the following properties are equivalent:*

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1) A family $\{F_\alpha \mid \alpha \in A\}$ of subsets of X is *monotone increasing* (resp. *monotone decreasing*) if A is well ordered and $F_\alpha \supset F_\beta$ (resp. $F_\alpha \subset F_\beta$) for each $\alpha \geq \beta$; $\alpha, \beta \in A$.

2) A topological property P is said to be *weakly hereditary* if every closed subspace of X has the property P whenever X has the property P.

3) See E. Michael [4].

(1) *X is a* $\mathfrak{B}$*-space.*

(2) *Every monotone increasing open covering of X has a cushioned open covering of X as a refinement.*

(3) *Every monotone increasing open covering of X has a σ-cushioned open covering of X as a refinement.*

Proof. (1) *implies* (2). Let $\mathfrak{H}=\{H_\alpha \mid \alpha<\lambda\}$ be an arbitrary monotone increasing open covering of X where we may assume that λ is a limit ordinal number. Let $G_\alpha=\bigcup_{\beta<\alpha} H_\beta$ for $\alpha<\lambda$, then it is easily seen that $\mathfrak{G}=\{G_\alpha \mid \alpha<\lambda\}$ is a monotone increasing open covering of X such that $G_\alpha \subset H_\alpha$ for each $\alpha \in [0, \lambda)$.

Furthermore we shall show the following:

$\bigcup_{\beta<\alpha} G_\beta = G_\alpha$ for any limit ordinal number $\alpha<\lambda$.

Since $\bigcup_{\beta<\alpha} G_\beta \subset G_\alpha$ is trivial, let x be any element of $G_\alpha=\bigcup_{\beta<\alpha} H_\beta$. Then $x \in H_\beta$ for some $\beta<\alpha$, and hence, $x \in H_\beta \subset G_{\beta+1}$, where $\beta+1<\alpha$ follows the fact that α is a limit ordinal number, that is, $x \in \bigcup_{\beta<\alpha} G_\beta$.

For this monotone increasing open covering $\{G_\alpha \mid \alpha<\lambda\}$, there exists a monotone increasing open covering $\mathfrak{U}=\{U_\alpha \mid \alpha<\lambda\}$ such that $\overline{U_\alpha} \subset G_\alpha$ for each $\alpha<\lambda$. We shall show that $\mathfrak{U}$ is a cushioned refinement of $\mathfrak{G}$, and hence, of $\mathfrak{H}$.

For this purpose, let A be an arbitrary subset of $[0, \lambda)$. If A has a maximal element or A is cofinal[4] in $[0, \lambda)$, $\overline{\bigcup_{\alpha\in A} U_\alpha} \subset \bigcup_{\alpha\in A} G_\alpha$ is trivial. Therefore we may assume that there exists a supremum α_0 of A in $[0, \lambda)$ and $\alpha_0 \notin A$. Then $\overline{\bigcup_{\alpha\in A} U_\alpha} \subset \overline{U_{\alpha_0}} \subset G_{\alpha_0} = \bigcup_{\alpha\in A} G_\alpha$ because the last inclusion follows the limit ordinality of α_0.

From the above, $\{U_\alpha \mid \alpha<\lambda\}$ is a monotone increasing open covering of X and a cushioned refinement of $\{H_\alpha \mid \alpha<\lambda\}$.

(2) *implies* (3). It is trivial.

(3) *implies* (1). Let $\mathfrak{G}=\{G_\alpha \mid \alpha<\lambda\}$ be any monotone increasing open covering of X, then there exists a σ-cushioned open covering $\mathfrak{B}=\bigcup_{i=1}^{\infty} \mathfrak{B}_i$ where we may assume that $\mathfrak{B}_i=\{B_\alpha^i \mid \alpha<\lambda\}$ and $\overline{\bigcup_{\alpha\in A} B_\alpha^i} \subset \bigcup_{\alpha\in A} G_\alpha$ for any subset A of $[0, \lambda)$ and each $i=1, 2, \cdots$. Let $B_i=\bigcup\{B_\alpha^i \mid \alpha<\lambda\}$, then the countable paracompactness of X being clear (by the lemma), we have a locally finite countable open covering $\mathfrak{W}=\{W_i \mid i=1, 2, \cdots\}$ of X such that $\overline{W_i} \subset B_i$ for each $i=1, 2, \cdots$. It will be sufficient to find a monotone increasing open covering $\mathfrak{U}=\{U_\alpha \mid \alpha<\lambda\}$ of X such that $G_\alpha \supset \overline{U_\alpha}$ for each $\alpha \in [0, \lambda)$.

For this purpose we put $U_\alpha=\bigcup_{i=1}^{\infty}\left\{\left(\bigcup_{\beta\leq\alpha} B_\beta^i\right) \cap W_i\right\}$ for each $\alpha<\lambda$.

4) A is said to be *cofinal in* $[0, \lambda)$ if, for each $\alpha \in [0, \lambda)$, there exists some element β of A such that $\alpha \leq \beta$.

(I) $\{U_\alpha \mid \alpha<\lambda\}$ *is a monotone increasing open covering of* X. It is clear.

(II) $\overline{U}_\alpha \subset G_\alpha$ *for each* $\alpha<\lambda$. From the local finiteness of $\{W_i \mid i\}$, $\left\{\left(\bigcup_{\beta\le\alpha} B_\beta^i\right)\cap W_i \mid i\right\}$ is locally finite, and hence,

$$\overline{U}_\alpha=\bigcup_{i=1}^{\infty}\overline{\left(\bigcup_{\beta\le\alpha} B_\beta^i\right)\cap W_i}\subset\bigcup_{i=1}^{\infty}\overline{\bigcup_{\beta\le\alpha} B_\beta^i}\subseteq\bigcup_{i=1}^{\infty}\left(\bigcup_{\beta\le\alpha} G_\beta\right)=\bigcup_{\beta\le\alpha} G_\beta=G_\alpha.$$

From (I) and (II), we complete the proof of (3)→(1).

Theorem 3. *In order that a topological space X be a $\mathfrak{B}$-space it is necessary and sufficient that every monotone increasing open covering $\{G_\alpha \mid \alpha<\lambda\}$ of X has the open covering $\mathfrak{U}=\bigcup_{i=1}^{\infty}\mathfrak{U}$ of X such that $\mathfrak{U}_i=\{U_\alpha^i \mid \alpha<\lambda\}$ is monotone increasing and $\overline{U}_\alpha^i\subset G_\alpha$ for each $\alpha<\lambda$ and $i=1, 2, \cdots$.*

Proof. *Necessity.* It is trivial.

Sufficiency. Let $\mathfrak{H}=\{H_\alpha \mid \alpha<\lambda\}$ be any monotone increasing open covering of X. Under the same discussion of the proof of [Theorem 2: (1)→(2)], we have the monotone increasing open covering $\mathfrak{G}=\{G_\alpha \mid \alpha<\lambda\}$ of X such that $G_\alpha\subset H_\alpha$ for each $\alpha<\lambda$ and, if $\alpha(\in[0, \lambda))$ is a limit ordinal number, then $\bigcup_{\beta<\alpha} G_\beta=G_\alpha$.

For this monotone increasing open covering $\mathfrak{G}$, there exists an open covering $\mathfrak{U}=\bigcup_{i=1}^{\infty}\mathfrak{U}_i$ of X such that $\mathfrak{U}_i=\{U_\alpha^i \mid \alpha<\lambda\}$ is monotone increasing and $\overline{U_\alpha^i}\subset G_\alpha$ for each $\alpha<\lambda$, each $i=1, 2, \cdots$. From Theorem 2, it will be sufficient to show only the fact that $\mathfrak{U}_i$ is cushioned in $\mathfrak{G}$ for each $i=1, 2, \cdots$. On the other hand, it is trivial by the discussion of Theorem 2: (1)→(2), and hence it completes the proof of Theorem 3.

Let $X_1, X_2, \cdots$ be topological spaces, then it is the interesting problem that $\prod_{i=1}^{n} X_i$ has the topological property P for each n, then $\prod_{i=1}^{\infty} X_i$ has the property P or not. It is known if P is the following classes: (1) Perfectly normal spaces (M. Katětov [3]), (2) perfectly normal and paracompact spaces (A. Okuyama [6]) and (3) perfectly normal and Lindelöf spaces (E. Michael [5]). Lastly we shall show the following:

Theorem 4. *Let $X_1, X_2, \cdots$, be topological spaces. If $\prod_{i=1}^{n} X_i$ is perfectly normal and the $\mathfrak{B}$-space for every $n=1, 2, \cdots$, then $\prod_{i=1}^{\infty} X_i$ is perfectly normal and the $\mathfrak{B}$-space.*

Proof. $X=\prod_{i=1}^{\infty} X_i$ is trivially perfectly normal (see M. Katětov [3]). Let $\mathfrak{U}=\{U_\alpha \mid \alpha<\lambda\}$ be an arbitrary increasing open covering of X, and

$U_\alpha^n = \cup\{U \mid U\colon$ open in $\prod_{i=1}^n X_i$, $U\times \prod_{i=n+1}^\infty X_i \subset U_\alpha$, then it is trivial that $U_\alpha = \bigcup_{n=1}^\infty \left\{U_\alpha^n \times \prod_{i=n+1}^\infty X_i\right\}$ for each $\alpha<\lambda$ and $\{U_\alpha^n \mid \alpha<\lambda\}$ is an increasing open covering of $U^n = \bigcup_{\alpha<\lambda} U_\alpha^n$, for every $n=1,2,\cdots$. From the perfect normality of $\prod_{i=1}^n X_i$, $U_n = \bigcup_{m=1}^\infty G_m^n$ for some open sets G_m^n in $\prod_{i=1}^n X_i$ and $\overline{G_m^n} \subset G_{m+1}^n$. Furthermore $\overline{G_m^n}$ being a $\mathfrak{B}$-space for each m, $\{U_n^\alpha \cap \overline{G_m^n} \mid \alpha<\lambda\}$ has the monotone increasing open (in $\overline{G_m^n}$) covering $\mathfrak{V}_m^n = \{V_{n,m}^\alpha \mid \alpha<\lambda\}$ of $\overline{G_m^n}$ such that $\overline{V_{n,m}^\alpha}$ $\left(\text{where closure in } \overline{G_m^n} \text{ and hence in } \prod_{i=1}^n X_i\right) \subset U_n^\alpha \cap \overline{G_m^n} \subset U_n^\alpha$. If we let $\mathfrak{W}_m^n = \left\{W_{n,m}^\alpha = (V_{n,m}^\alpha \cap G_m^n) \times \prod_{i=n+1}^\infty X_i \mid \alpha<\lambda\right\}$, then it is trivial that $\mathfrak{W}_m^n$ is an open (in X) covering a of $G_m^n \times \prod_{i=n+1}^\infty X_i$ such that

$$\overline{W_{n,m}^\alpha} = \overline{V_{n,m}^\alpha \cap G_m^n} \times \prod_{i=n+1}^\infty X_i \subset \overline{V_{n,m}^\alpha} \times \prod_{i=n+1}^\infty X_i \subset U_n^\alpha \times \prod_{i=n+1}^\infty X_i \subset U_\alpha$$

Next, we shall show that $\mathfrak{W}_m^n$ is an increasing open collection for every n, m and $\bigcup_{n,m=1}^\infty \mathfrak{W}_m^n$ is an open covering of X. These statements are easily seen and therefore we complete the proof of Theorem 4 by Theorem 3.

Remark. (1) Clearly, $\mathfrak{B}$-spaces are countably paracompact spaces. But the converse is not true (see Y. Yasui [8]).

(2) In the definition of a $\mathfrak{B}$-space, we can not drop the condition that $\{G_\alpha \mid \alpha\in A\}$ is a monotone decreasing family, that is, there exists a space X such that X is not a $\mathfrak{B}$-space but every monotone decreasing closed collection $\{F_\alpha \mid \alpha \in A\}$ with vacuous intersection has the open collection $\{G_\alpha \mid \alpha\in A\}$ with the property that $\bigcap_{\alpha\in A} \overline{G_\alpha}$ is empty and $G_\alpha \supset F_\alpha$ for each $\alpha\in A$ (see Y. Yasui [8]).

(3) The product space of $\mathfrak{B}$-space with $\mathfrak{B}$-space is not necessarily $\mathfrak{B}$-space (see Y. Yasui [8]).

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