15. *Some Investigations on Many Valued Logics*

By Nobuyoshi MOTOHASHI

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In their book [1], Chang and Keisler developed a study of theories of models with truth values in compact Hausdorff spaces. One of the main reasons why they required some topological properties is that a basic tool used there is the compactness theorem. But, we can develop a study of model theories on logics without compactness properties.

In this paper, we shall study logics with truth values in some set X, in which we don't assume any topological properties and some theories of models on these logics by the method developed in [2]–[4].

Many valued logic $\mathcal{L}_1=\mathcal{L}_1(X, C, Q, \underset{\sim}{1})$. Let X be a non empty *countable* set with a designated element $\underset{\sim}{1} \in X$, X^* be the set of all non empty subsets of X, C be a set of finitary functions on X, and Q be a set of unary functions on X^* to X. Then by the usual manner, we can construct a many valued logic $\mathcal{L}_1=\mathcal{L}_1(X, C, Q, \underset{\sim}{1})$ with equality $\simeq$ except that we admit the following role of quantifiers; if $q \in Q$ and Σ is a set of formulas in $\mathcal{L}_1$ such that $1 \leqq \bar{\bar{\Sigma}} \leqq \bar{\bar{X}}$, then $q(\Sigma)$ is a formula in $\mathcal{L}_1$. Also, we can define the semantical notions such as $\mathcal{L}_1$-structure $\mathcal{A}$, and assignment r in $\mathcal{A}$, $\sigma[\mathcal{A}, r] \in X$ for any formula σ in $\mathcal{L}_1$.

If σ and τ are formulas in $\mathcal{L}_1$, "σ is a consequence of τ" (written by $\tau \models \sigma$) means that $\tau[\mathcal{A}, r]=\underset{\sim}{1}$ implies $\sigma[\mathcal{A}, r]=\underset{\sim}{1}$ for any $\mathcal{A}, r$ and "σ is valid" (written by $\models \sigma$) means that $\sigma[\mathcal{A}, r]=\underset{\sim}{1}$ for any $\mathcal{A}, r$.

Two valued logic $\mathcal{L}=\mathcal{L}(\mathcal{L}_1)$ as a metalogic of $\mathcal{L}_1$. $\mathcal{L}=\mathcal{L}(\mathcal{L}_1)$ can be defined from $\mathcal{L}_1$ by the following rules:

(1) If σ is a formula in $\mathcal{L}_1$ and $x \in X$, then (σ, x) is formula in $\mathcal{L}$. (if σ is an atomic formula in $\mathcal{L}_1$, (σ, x) is called an *atomic* formula in $\mathcal{L}$).

(2) Usual closure under two valued logical operations $\neg, \wedge, \vee, \forall, \exists$ except that $\wedge$ and $\vee$ are only applied to sets Φ of formulas such that $1 \leqq \bar{\bar{\Phi}} \leqq \bar{\bar{X}}$.

If a formula θ in $\mathcal{L}$ can be constructed from only atomic formulas in $\mathcal{L}$ by applying $\neg, \wedge, \vee, \forall, \exists$, then θ is called *normal*. For any $\mathcal{L}_1$-structure $\mathcal{A}$, any assignment r in $\mathcal{A}$ and any formula θ in $\mathcal{L}$, we can define the satisfaction relation $\mathcal{A} \models \theta[r]$ by the usual method. Let $FM(\mathcal{L})$ and $PFM(\mathcal{L})$ be the set of formulas in $\mathcal{L}$ and the set of valid formulas in $\mathcal{L}$. Then clearly these $FM(\mathcal{L})$ and $PFM(\mathcal{L})$ satisfy the conditions stated in [2].

In particular, $\tau \models \sigma \Longleftrightarrow \vdash_L (\tau, \underset{\sim}{1}) \supset (\sigma, \underset{\sim}{1})$.

Logic L^I. From $\mathcal{L}$, we construct the logic L^I by the method stated in [2]. If a formula F in L^I is constructed from formulas $\theta^1, \theta^2, I_i(x, y)$, where θ is a normal formula in $\mathcal{L}$, by applying $\neg, \wedge, \vee, \forall, \exists$, we say that F is *normal*.

The semantics and syntax for L^I are easily defined by the methods in [3], [4]. By M, we shall denote L^I-structures.

Now, we shall consider the following three statements with respect to $\mathcal{L}_1, \mathcal{L}, L^I$.

(I) (Completeness theorem for L^I). For any formula F in L^I, if F is not provable in L^I, then there is a countable L^I-structure M and an assignment r in M such that "not $M \models F[r]$" holds.

(II) (Normal form theorem for $\mathcal{L}$). Every formula in $\mathcal{L}$ is equivalent to a normal formula in $\mathcal{L}$.

(III) (Reduction theorem of $\mathcal{L}$ to $\mathcal{L}_1$). There is a mapping $*$ from the set of formulas in $\mathcal{L}$ to that of $\mathcal{L}_1$ such that

$$\vdash_L \theta = (\theta^*, \underset{\sim}{1}) \qquad \text{for any formula } \theta \text{ in } \mathcal{L}.$$

Theorem 1. (I) *is equivalent to* (II) i.e. *Completeness theorem for L^I and Normal form theorem for $\mathcal{L}$ are equivalent.*

Proof (Outline). *Assume* (I). Let θ be an arbitrary formula in $\mathcal{L}$ and Ψ be the primitive set saying that I_0 is an isomorphism. Then $\Delta(\Psi)$ is the set of normal formulas in $\mathcal{L}$. Since θ is preserved under isomorphism, θ is equivalent to a formula in $\Delta(\Psi)$, i.e. a normal formula, by (I) and the results in [3]. Hence (II) holds.

Assume (II). Then every formula in $\mathcal{L}$ is equivalent to a normal formula in $\mathcal{L}$. Hence every formula in L^I is equivalent to a normal formula in L^I. But completeness theorem for normal formulas in L^I is easily proved by the straight-forward generalization of the standard method using the countability of X. This means that (I) holds. Q.E.D.

In order to state a sufficient condition for (I)((II)), we shall define a new language L^X.

L^X has only one unary predicate variable α, individual constants e_x for each $x \in X$, k-ary predicate constants R_s for each k-ary relations S on X and $\neg, \wedge, \vee$.

By using these symbols, we can define formulas and the notion that $X \models D[Y]$ (Y satisfies D in X) for each $Y \subseteq X$, each formula D in L^X, in the usual manner.

A subfamily $z \subseteq X^*$ is *definable* in X if there is a formula D in L^X such that $z = \{Y ; Y \in X^*, X \models D[Y]\}$, a quantifier q is *definable* in X if $\{Y ; Y \in X^*, q(Y) = x\}$ is definable in X for each $x \in X$ and Q is *definable* in X if every $q \in Q$ is definable in X.

Theorem 2. *If Q is definable in X, then* (II) *holds* (*hence* (I) *also*).

Proof (Outline). Assume Q is definable in X. It's sufficient to prove the for any formula σ in $\mathcal{L}_1$ and $x \in X$, there is a normal formula θ in $\mathcal{L}$ such that $\vdash_L (\sigma, x) \equiv \theta$.

This can be proved by the induction on σ. Q.E.D.

Theorem 3. *If C and Q satisfy the following conditions* (a)–(c) *then* (III) *holds.*

(a) *For any $x \neq \underset{\sim}{1}$ in X, there is a $c_x \in C$ such that for any $y \in X$,*

$$c_x(y) = \underset{\sim}{1} \Longleftrightarrow x = y$$

(b) *There is a $c_\neg \in C$ such that for any $x \in X$,*

$$c_\neg(x) = \underset{\sim}{1} \Longleftrightarrow x \neq \underset{\sim}{1}.$$

(c) *There are $q_\forall, q_\exists \in Q$ such that for any $Y \in X^*$*

$$q_\forall(Y) = \underset{\sim}{1} \Longleftrightarrow Y = \{\underset{\sim}{1}\}$$

$$q_\exists(Y) = \underset{\sim}{1} \Longleftrightarrow \underset{\sim}{1} \in Y.$$

Proof (Outline). Define θ^* by $(\sigma, \underset{\sim}{1})^* = \sigma$, $(\sigma, x)^* = c_x(\sigma)(x \neq \underset{\sim}{1})$

$(\neg\theta)^* = c_\neg(\theta^*)$, $(\wedge\Phi)^* = q_\forall(\{\varphi^*\,;\ \varphi \in \Phi\})$, $(\vee\Phi)^* = q_\exists(\{\varphi^*\,;\ \varphi \in \Phi\})$

$((\forall v)\theta(v))^* = (q_\forall v)\theta^*(v)$, $((\exists v)\theta(v))^* = (q_\exists v)\theta^*(v)$. Q.E.D.

So, if $\mathcal{L}_1 = \mathcal{L}_1(X, C, Q, \underset{\sim}{1})$ satisfies (a)–(c) in Theorem 3 and Q is definable in X, then we can assert the following statements:

(I) Craig-Lyndon like interpolation holds in L_1.

(II) All the preservation theorems stated in [4] hold in $\mathcal{L}_1$.

(III) $\mathcal{L}_1$ is functionally complete in the following sense; every quantifier definable in X and every connectives are expressible in $\mathcal{L}_1$.

Remark 1. *If X is finite, then every quantifier on X is definable in X.*

Remark 2. *If $\overline{\overline{X}}$ is ω, then the cardinality of the set of quantifiers on X is 2^{2^ω} and the cardinality of the set of definable quantifiers on X is 2^ω. Hence there are many quantifiers which are not definable in X.*

Remark 3. *The usual two valued logic and the three valued logic of R. R. Rockingham Gill in* [5] *satisfy the above conditions with slight modifications. Hence our results imply the results in* [5].

Remark 4. *If $\overline{\overline{X}}$ is ω and $\mathcal{L}_1$ satisfies the above conditions, then we can assume every connective belongs to C by* (III).

Therefore any topology on X cannot make $\mathcal{L}_1$ continuous logic. Hence our results are not included in those of [1].

References

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