54. *Necessary and Sufficient Conditions for Countable Compactness of Product Spaces*

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In this paper a product space is the topological product of two spaces.

We know in [1] the necessary and sufficient conditions for normality of product spaces. The conditions are obtained by describing the normality of product space on its factor spaces. The idea of such the description has been used explicitly or implicitly in some literatures. In fact, the description way is useful and sometimes necessary in the study of product spaces as [2], [3] and [4] also show.

In this paper we intend to present another application of the description method by giving the conditions stated in the title. Since 1953 when Terasaka [13] and Novák [11], answering a question posed by Čech in 1938, showed that the product of two countably compact spaces need not be countably compact, the condition for countable compactness of a product space has been searched and several sufficient ones were obtained. We shall now show complete conditions in Theorem 2. One of merits of the theorem will be seen in Corollary 6 to the theorem which generalizes a theorem of Frolík [8] and is proved fairly more simply than in [8]. Some topics on the closedness of projection maps of product spaces are also discussed.

Throughout this paper, unless otherwise stated, spaces are T_1 spaces.

We first recall some notations and results in [1] and [2] for later use. For a subset A of the product space $X\times Y$ we write $A[x]=\{y\,;(x,y)\in A\}$ for each point $x\in X$. Let $\mathfrak{F}=\{F_x\,;x\in Z\subset X\}$ be a family of subsets of Y with suffixes of points in X, then we write

$$\limsup_a \mathfrak{F}=\limsup_a F_x=\bigcap_{U\in\mathfrak{N}_a}\overline{\bigcup_{x\in U}F_x}$$

for any point $a\in X$, where $\mathfrak{N}_a$ is the neighbourhood system of a in X. We know [1] that, putting $F=\{(x,y)\,;x\in Z,y\in F_x\}\subset X\times Y$, we have

$$(*)\qquad \bar{F}[a]=\limsup_a F_x$$

for any $a\in X$; and that

$$(**)\qquad \limsup_a F_x=\limsup_a(\limsup_z F_x).$$

A space Y is *upper compact at a point* a of X if for any family $\mathfrak{F}=\{F_x\,;x\in Z\subset X\}$ of non-void subsets of Y with $a\in\bar{Z}-Z$ we have

$\lim\sup_{a} \mathfrak{F} \neq \emptyset$. Y is *upper compact at* X if Y is upper compact at every point of X. A point is a *cluster point* of a family of subsets if any neighborhood of the point meets infinitely many members of the family. We know [2]

Theorem 1. *A space Y is upper compact at a space X if and only if the projection map of the product $X \times Y$ to X is closed.*
(Though the Hausdorff separation axiom is assumed in [2], this theorem needs no separation axiom.)

It is immediately deduced from this theorem that Y is compact if and only if the projection of $X \times Y$ to X is closed for any X; and if Y is $\mathfrak{m}$-compact for an infinite cardinal number $\mathfrak{m}$ and if the character of every point of X is $\leq \mathfrak{m}$, then the projection is closed [2]. Some other known results on closedness of the projection are also direct consequences of this theorem.

We denote by $\mathfrak{Q}$ the class consisting of all X such that the projection of $X \times Y$ to X is closed for any space Y in a class $\mathfrak{R}$ of spaces which is closed-hereditary and closed under a continuous and open map, i.e., if $S \in \mathfrak{R}$ and T is a closed subspace of S, then $T \in \mathfrak{R}$ and $f(S) \in \mathfrak{R}$ for any continuous and open map f. We can consider, for instance, as $\mathfrak{R}$ the class of all countably compact spaces, or locally compact spaces, or Fréchet spaces (as for the definition or properties of the last spaces, see below or [6] and [7]). The following proposition unifies and generalizes some results of Isiwata [9], which can be proved in a similar way to his proof of Theorem 1.3 in [9].

Proposition 1. *Suppose a T_2 space X has the property*

(α) *For any $Y \in \mathfrak{R}$ and for every accumulation point p of a subset A of X there is a subset B of A such that p is an accumulation point of B and $\bar{B} \times Y \in \mathfrak{R}$.*

Then $X \in \mathfrak{Q}$ if and only if any subset, belonging to $\mathfrak{R}$, of X is closed.
(For "if" part of this proposition X need not be T_2.)

A topological space X is a *Fréchet space* if the closure of any subset A of X is the set of limits of sequences in A. We denote by $\mathfrak{Q}_0$ $\mathfrak{Q}$ for $\mathfrak{R}_0$ which is the class of all countably compact spaces ($\mathfrak{Q}_0$ is $\mathfrak{R}$ in Isiwata's notation in [9]).

Corollary 1 (Isiwata [9]). *A Fréchet space X with unique sequential limits belongs to $\mathfrak{Q}_0$.*

Proof. Suppose $p \in X$ is an accumulation point of a subset A of X, then there is a sequence $\{x_n\}$ converging to p in A; since $\overline{\{x_n\}}$ is compact, $\overline{\{x_n\}} \times Y \in \mathfrak{R}_0$ for any $Y \in \mathfrak{R}_0$, namely X satisfies (α) in Proposition 1. Since any countably compact subset is closed in a Fréchet space with unique sequential limits (Proposition 5.4, [7]), $X \in \mathfrak{Q}_0$.

Corollary 2 (Isiwata [9]). *Let X be a locally compact T_2 space.*

$X \in \mathfrak{Q}_0$ *if and only if any countably compact subset of* X *is closed.*

Proof. Suppose $p \in X$ is an accumulation point of subset A of X, and U a compact neighbourhood of p, then p is an accumulation point of $B = A \cap U$ and $\bar{B}$ is compact, and so X satisfies the condition (α) in Proposition 1.

Fleischer and Franklin [5] present without proof the following better result than our Corollary 1 above. We shall prove it because it will be used later. (The statement on p. 78 in [5] "if the projection of $X \times S_1$ on S_1 is closed, then X is sequentially compact" is obviously incorrect because we know the existence of a compact space which is not sequentially compact. According to the communication of Franklin to the author, the assumption that X is sequential should be added. Then this is an immediate consequence from his Proposition 1.10 in [6] (cf. also the proof of Proposition 5.6, [7]) and the result stated in the last part on p. 77 in [5]).

A subset F of a topological space X is *sequentially closed* if a sequence in F converges to a point, then the point is in F, and X is a *sequential space* if each sequentially closed subset of X is closed [6].

Proposition 2 (Fleischer and Franklin [5]). *Every sequential space belongs to* $\mathfrak{Q}_0$.

Proof. Let X be sequential, Y countably compact, H a closed subset of $X \times Y$, and let $\{x_n\}$ a sequence in $F = pr_X(H)$ converging to a point p. Let us show p belongs to F. Consider the family $\mathfrak{F} = \{H[x_n]\,;\ n = 1, 2, \cdots\}$ and put $K = \{(x_n, y)\,;\ y \in H[x_n], n = 1, 2, \cdots\}$, then since a number of different members of $\{\bigcup_{x_n \in U} H[x_n]\,;\ U \in \mathfrak{N}_p\}$ is at most countable, we have

$$\begin{aligned} \emptyset \neq \limsup_p \mathfrak{F} &= \bigcap_{U \in \mathfrak{N}_p} \overline{\bigcup_{x_n \in U} H[x_n]} \\ &= \bar{K}[p] \subset H[p] \end{aligned}$$

by $(*)$, and $p \in F$.

Definition. A space Y is said to be *weakly upper compact at a space* X if for any family $\mathfrak{F} = \{F_x\,;\ x \in Z \subset X\}$ of non-void subsets of Y with Z having cluster points in X we have $\limsup_a \mathfrak{F} \neq \emptyset$ for some cluster point a of Z.

Remark. We may assume in this definition that F_x is a point of Y, and that Z is a set of countably many different points of X if X is countably compact (or a countable discrete subspace of X if X is additionally T_2). If Y is upper compact at X, then it is weakly upper compact at X. The next theorem may be interesting also in comparison with Theorem 1.

Theorem 2. *The following are equivalent for countably compact spaces* X *and* Y.

(i) $X \times Y$ *is countably compact.*

(ii) *The projection of any closed subset in $X\times Y$ to X is countably compact.*

(iii) *Y is weakly upper compact at X.*

Proof. (i)$\Rightarrow$(ii). If $X\times Y$ is countably compact, then a closed subset in $X\times Y$ is, and so its continuous image is countably compact.

(ii)$\Rightarrow$(iii). Given a family $\mathfrak{F}=\{F_x\,;\,x\in Z\subset X\}$ of non-void subsets of Y with Z having cluster points in X, then $F=\{(a,y)\,;\,a\in X, y\in \limsup_a \mathfrak{F}\}$ is a closed set in $X\times Y$ by (*) and (**); in fact, for any $p\in X$, $\bar{F}[p]=\limsup_p(\limsup_a \mathfrak{F})=\limsup_p \mathfrak{F}=F[p]$. Consequently, $G=pr(F)=\{a\,;\,\limsup_a \mathfrak{F}\neq\emptyset\}$ is countably compact; since $G\supset Z$, G includes a cluster point c of Z, namely, $\limsup_c \mathfrak{F}\neq\emptyset$.

(iii)$\Rightarrow$(i). Let $D=\{(x_n,y_n)\,;\,n=1,2,\cdots\}$ be a sequence of different points in $X\times Y$. If $Z=\{x_n\,;\,n=1,2,\cdots\}$ is a finite set, then there is an infinite set $\{n_i\,;\,i=1,2,\cdots\}$ of natural numbers such that $x_{n_i}=x_0$ a fixed point in X for all i and $\{y_{n_i}\,;\,i=1,2,\cdots\}$ is infinite. Since Y is countably compact, $\{y_{n_i}\}$ has a cluster point $y_0\cdot(x_0,y_0)$ is a cluster point of D. Suppose next Z is infinite. (1) If for some $p\in X$, Z contains an infinite subset $\{x_{n_i}\,;\,i=1,2,\cdots\}$ such that, for any neighborhood U of p, $\{i\,;\,x_{n_i}\notin U\}$ is a finite set, then the sequence $\{x_{n_i}\}$ converges to p. Let q be a point of Y which is a cluster point of $\{y_{n_i}\}$ if $\{y_{n_i}\}$ is an infinite set, or which coincides with y_{n_i} for infinitely many i if $\{y_{n_i}\}$ is a finite set. (p,q) is a cluster point of D. (2) If, for any point p of X and for any infinite subset $\{x_{n_i}\}$ of Z, there is a neighborhood U of p such that $\{i\,;\,x_{n_i}\notin U\}$ is an infinite set, then there is an infinite discrete subspace $Z'=\{x_{m_k}\,;\,k=1,2,\cdots\}$ of Z. Put $\mathfrak{F}=\{y_x\,;\,x\in Z',(x,y_x)\in D\}$, then, since Y is weakly upper compact at X, $\limsup_a \mathfrak{F}$ includes a point b of Y for some cluster point a of Z' in X. Since a does not belong to Z', (a,b) is a cluster point of D.

Corollary 1. *Let X and Y be countably compact. The product $X\times Y$ is countably compact if Y is upper compact at every cluster point of countably many points in X.*

Proof. Y is weakly upper compact at X.

Corollary 2 (Ryll-Nardzewski [12]). *Suppose X and Y are countably compact. If Y is upper compact at X, then $X\times Y$ is countably compact.*

Corollary 3 (Ryll-Nardzewski [12]). *Let $\mathfrak{m}$ be an infinite cardinal. If X is countably compact, if the character of every point of X is $\leq\mathfrak{m}$, and if Y is $\mathfrak{m}$-compact, then $X\times Y$ is countably compact.*

Proof. Y is upper compact at X [2].

Corollary 4 (Franklin [6]). *The product of two countably compact spaces, one of which is sequential, is countably compact.*

Proof. Let X and Y be countably compact the former of which be sequential, then $pr: X\times Y\to X$ is closed by Proposition 2, so the projection of a closed subset of $X\times Y$ to X is countably compact, and $X\times Y$ is countably compact by Theorem 2.

We say a space to be *sequentially compact* if each sequence of points of the space contains a convergent subsequence. We have proved the following in the last part of the proof of Theorem 2.

Corollary 5 (Mrówka [10]). *The product of two countably compact spaces, one of which is sequentially compact, is also countably compact.*

Denote by $\mathfrak{C}$ the class of all spaces X such that, for every countably compact space $Y, X\times Y$ is countably compact. The next corollary generalizes a theorem of Frolík because the spaces are here not necessarily completely regular. And it is not required that the space K in the condition below is, as in [8], a Hausdorff compactification of X.

Corollary 6 (Frolík [8]). *A space X belongs to $\mathfrak{C}$ if and only if it satisfies the following condition: For any infinite discrete subspace N of X, there is a space K containing X as a subspace and belonging to $\mathfrak{C}$ such that $N\cup S$ is not countably compact for any subset S of $K-X$.*

Proof. If X belongs to $\mathfrak{C}$, then X itself plays the role of K in the condition. Suppose conversely X satisfies the condition. If X is not countably compact, then there is an infinite discrete closed subspace N of X and a space K stated in the condition. $\bar{N}-N\subset K-X$, where the bar means the closure in K, which contradicts the property of K because $\bar{N}$ is countably compact. Consequently X is countably compact. Let Y be any countably compact space, and let F a closed subset in $X\times Y$. Suppose $E=pr(F)$ is not countably compact in X, then it contains an infinite discrete (in E) subset N', and we have a space K in the condition. $H=\{(x,y)\in F\,;\, x\in N'\}$ is closed in $X\times Y$, and $pr(\tilde{H})$ is countably compact, where $\sim$ means the closure in the countably compact $K\times Y$; while $pr(\tilde{H})-N'\subset K-X$, which contradicts the property of K. Consequently E is countably compact and $X\times Y$ is countably compact by Theorem 2.

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