8. *Universal Sentences Preserved under Certain Extensions*

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The purpose of this paper is to characterize the universal sentences preserved under the formation of zero-element extensions. Here a zero-element extension is defined as follows.

Let $\mathfrak{A}=\langle A, \{f_\xi \mid \xi<\rho\}, \{r_\eta \mid \eta<\sigma\}\rangle$ and $\mathfrak{A}^*=\langle A^*, \{f_\xi^* \mid \xi<\rho\}, \{r_\eta^* \mid \eta<\sigma\}\rangle$ be structures of the same similarity type, where A, A^* are domains, f_ξ, f_ξ^* are $n(\xi)$-ary operations, and r_η, r_η^* are $m(\eta)$-ary relations. $\mathfrak{A}^*$ is called a *zero-element extension* of $\mathfrak{A}$ if the following three conditions hold:

(1) A^* consists of all elements in A and an element not contained in A which is denoted by o, i.e. $A^*=A\cup\{o\}$ and $o\notin A$;

(2) For any $\xi<\rho$ and any $a_1, \cdots, a_{n(\xi)}\in A^*$,

$$f_\xi^*(a_1, \cdots, a_{n(\xi)})=\begin{cases} a & \text{if } f_\xi(a_1, \cdots, a_{n(\xi)})=a, \\ o & \text{if at least one of } a_1, \cdots, a_{n(\xi)} \text{ is } o; \end{cases}$$

(3) For any $\eta<\sigma$ and any $a_1, \cdots, a_{m(\eta)}\in A^*$, $r_\eta^*(a_1, \cdots, a_{m(\eta)})$ if and only if either $r_\eta(a_1, \cdots, a_{m(\eta)})$ or $a_1=\cdots=a_{m(\eta)}=o$.

Each of the well-known preservation theorems asserts that a sentence is preserved under a given algebraic construction (or constructions) if and only if it is equivalent to a sentence having certain formal properties which depend chiefly on occurrences of logical symbols. However, the formal properties of sentences, which appear in our discussion, depend largely on occurrences of individual variables as well as occurrences of logical symbols.

Let L be a first-order language with or without equality. A structure $\mathfrak{A}$ of the similarity type corresponding to L is simply called a structure for L. The domain of $\mathfrak{A}$ is denoted by $D[\mathfrak{A}]$. Let Φ be any formula of L which contains at most some of the distinct variables $x_1, \cdots, x_n$ as free variables, and let $a_1, \cdots, a_n$ be elements in $D[\mathfrak{A}]$. Then we write $\mathfrak{A}\models\Phi[a_1/x_1, \cdots, a_n/x_n]$, if $a_1, \cdots, a_n$ satisfy Φ in $\mathfrak{A}$ when the free variables $x_1, \cdots, x_n$ are assigned the values $a_1, \cdots, a_n$ respectively. If $\mathfrak{A}\models\Phi[a_1/x_1, \cdots, a_n/x_n]$ holds for any elements $a_1, \cdots, a_n$ in $D[\mathfrak{A}]$, we say that Φ is valid in $\mathfrak{A}$, and we write $\mathfrak{A}\models\Phi$. If $\mathfrak{A}\models\Phi$ holds for every structure $\mathfrak{A}$ for L, we write $\models\Phi$. Let Γ be a sentence or a set of sentences of L. A structure $\mathfrak{A}$ for L is called a model of Γ if $\mathfrak{A}\models\Gamma$ or $\mathfrak{A}\models\Phi$ for every Φ in Γ. We denote by $\mathcal{M}(\Gamma)$ the class of all models of Γ. If $\mathcal{M}(\Gamma)$ is not empty, we say that Γ is *satisfiable*. Furthermore, let Δ be a sentence or a set of sentences of L. If $\mathcal{M}(\Gamma)\subseteq\mathcal{M}(\Delta)$, we write $\Gamma\models\Delta$. If $\Gamma\models\Delta$ and $\Delta\models\Gamma$, we say that Γ is equivalent to Δ, and we write $\Gamma\Leftrightarrow\Delta$. Hereafter, "structure" will mean "structure for L", "formula" will mean "formula of L", and so on.

A formula of the form $r(t_1, \cdots, t_m)$ is called an atomic formula, where r is an m-ary relation symbol and $t_1, \cdots, t_m$ are terms. If L has the equality symbol $\equiv$, a formula of the form $t\equiv u$ is also called an atomic formula, where t, u are terms. The terms $t_1, \cdots, t_m$ and t, u are called *principal terms* of the atomic formulas $r(t_1, \cdots, t_m)$ and $t\equiv u$ respectively.

Let Θ be an atomic formula. If a variable y occurs in all principal terms of Θ, we say that *the atomic formula Θ is regular with respect to y*. If y occurs in Θ but it does not occur in at least one principal term of Θ, we say that *the formula $\neg\Theta$, the negation of the atomic formula, is regular with respect to y*. A formula is called a *basic formula*, if it is an atomic formula or the negation of an atomic formula.

Lemma 1. *Let Θ be a basic formula which contains at most some of the distinct variables $y, x_1, \cdots, x_n$. And let $\mathfrak{A}^*$ be a zero-element extension of a structure $\mathfrak{A}$, where $D[\mathfrak{A}^*]=D[\mathfrak{A}]\cup\{o\}$. Moreover let $a_1, \cdots, a_n$ be elements in $D[\mathfrak{A}]$. Then the following hold:*

(i) *Assume that Θ is regular with respect to y. Then*

$$\mathfrak{A}^*\models\Theta[o/y, a_1/x_1, \cdots, a_n/x_n].$$

(ii) *Assume that y occurs in Θ and Θ is not regular with respect to y. Then*

$$\mathfrak{A}^*\models\neg\Theta[o/y, a_1/x_1, \cdots, a_n/x_n].$$

Proof of (i). First suppose Θ is atomic. Then y occurs in all principal terms of Θ. Hence we have

$$\mathfrak{A}^*\models\Theta[o/y, a_1/x_1, \cdots, a_n/x_n].$$

Next suppose Θ is the negation of an atomic formula Ψ. Then y occurs in Ψ but it does not occur in at least one principal term of Ψ. Hence

$$\mathfrak{A}^*\models\neg\Psi[o/y, a_1/x_1, \cdots, a_n/x_n]$$

and hence

$$\mathfrak{A}^*\models\Theta[o/y, a_1/x_1, \cdots, a_n/x_n].$$

Proof of (ii). Let Ψ be a basic formula such that $\models\Psi\leftrightarrow\neg\Theta$. Then Ψ is regular with respect to y. Hence by (i) of this lemma,

$$\mathfrak{A}^*\models\Psi[o/y, a_1/x_1, \cdots, a_n/x_n].$$

Therefore we have

$$\mathfrak{A}^*\models\neg\Theta[o/y, a_1/x_1, \cdots, a_n/x_n].$$

This completes the proof.

A sentence of the form $\forall x_1\cdots\forall x_n(\Theta_1\vee\cdots\vee\Theta_m)$ is called a *disjunctive universal sentence*, where $\Theta_1, \cdots, \Theta_m$ are basic formulas (and $x_1, \cdots, x_n$ are distinct variables). Let Φ be a formula which contains at most some of the variables $x_1, \cdots, x_n$ as free or bound variables. Let X be a subset of $\{x_1, \cdots, x_n\}$ and let y be a variable not contained in $\{x_1, \cdots, x_n\}$. Then we denote by $\Phi^{X \to y}$ the formula formed from Φ by replacing all occurrences of the variables in X by the variable y. Note that if Φ is a universal sentence in prenex normal form, then $\Phi \models \Phi^{X \to y}$.

Let

$$\Phi = \forall x_1 \cdots \forall x_n (\Theta_1 \vee \cdots \vee \Theta_m)$$

be a disjunctive universal sentence, where $\Theta_1, \cdots, \Theta_m$ are basic formulas. We say that Φ is *regular* (or *weakly regular*), if the following condition $(*)$ holds for every non-empty (or unit) subset X of $\{x_1, \cdots, x_n\}$ and for a variable y not contained in $\{x_1, \cdots, x_n\}$:

$(*)$ If y occurs in $\Theta_1^{X \to y} \vee \cdots \vee \Theta_m^{X \to y}$, then there exists a formula $\Theta_i^{X \to y}$ $(1 \leqq i \leqq m)$ which is regular with respect to y.

Now we shall prove the following

Theorem 1. *Let Σ be a set of regular disjunctive universal sentences. Then $\mathcal{M}(\Sigma)$ is closed under the formation of zero-element extensions.*

Proof. If $\mathcal{M}(\Sigma)$ is empty, it is obvious that $\mathcal{M}(\Sigma)$ is closed under the formation of zero-element extensions. In the following, we assume that $\mathcal{M}(\Sigma)$ is not empty. Let $\mathfrak{A}$ be any structure in $\mathcal{M}(\Sigma)$, and let $\mathfrak{A}^*$ be a zero-element extension of $\mathfrak{A}$, where $D[\mathfrak{A}^*] = D[\mathfrak{A}] \cup \{o\}$. We shall prove that $\mathfrak{A}^* \in \mathcal{M}(\Sigma)$.

Now let

$$\Phi = \forall x_1 \cdots \forall x_n (\Theta_1 \vee \cdots \vee \Theta_m)$$

be any sentence in Σ, where $\Theta_1, \cdots, \Theta_m$ are basic formulas. And let $\langle a_1, \cdots, a_n \rangle$ be any sequence of elements in $D[\mathfrak{A}^*]$. To prove $\mathfrak{A}^* \in \mathcal{M}(\Sigma)$, it suffices to show that

$$(\sharp) \qquad \mathfrak{A}^* \models (\Theta_1 \vee \cdots \vee \Theta_m)[a_1/x_1, \cdots, a_n/x_n].$$

If o does not occur in $\langle a_1, \cdots, a_n \rangle$, we can immediately obtain $(\sharp)$. Now we assume that o occurs in $\langle a_1, \cdots, a_n \rangle$. Let X be the subset of $\{x_1, \cdots, x_n\}$ such that $x_i \in X$ if and only if $a_i = o$. And let y be a variable not contained in $\{x_1, \cdots, x_n\}$. Then it is clear that $(\sharp)$ is equivalent to

$$(\sharp\sharp) \qquad \mathfrak{A}^* \models (\Theta_1^{X \to y} \vee \cdots \vee \Theta_m^{X \to y})[o/y, a_1/x_1, \cdots, a_n/x_n].$$

If y does not occur in $\Theta_1^{X \to y} \vee \cdots \vee \Theta_m^{X \to y}$, it is obvious that $(\sharp\sharp)$ holds. Now suppose that y occurs in $\Theta_1^{X \to y} \vee \cdots \vee \Theta_m^{X \to y}$. Then by $(*)$, there exists a formula $\Theta_{i_0}^{X \to y}$ $(1 \leqq i_0 \leqq m)$ which is regular with respect to y. Hence by Lemma 1 (i), we have

$$\mathfrak{A}^* \models \Theta_{i_0}^{X \to y}[o/y, a_1/x_1, \cdots, a_n/x_n].$$

Therefore $(\sharp\sharp)$ holds also in this case. Hence we have $(\sharp)$, and hence $\mathfrak{A}^* \models \Phi$. Therefore we have that $\mathfrak{A}^* \in \mathcal{M}(\Sigma)$. This completes the proof.

In order to study the converse of Theorem 1, we need the notion of a reduced set of disjunctive universal sentences, which is substantially the same as in Definition 2 in [1; § 46].

Let Σ be a set of disjunctive universal sentences, and let

$$\Phi = \forall x_1 \cdots \forall x_n(\Theta_1 \vee \cdots \vee \Theta_m)$$

be a sentence in Σ, where $\Theta_1, \cdots, \Theta_m$ are basic formulas. We say that Φ is *reduced with respect to* Σ if $m=1$ or

$$\Sigma \models \forall x_1 \cdots \forall x_n(\Theta_1 \vee \cdots \vee \Theta_{i-1} \vee \Theta_{i+1} \vee \cdots \vee \Theta_m)$$

does not hold for any i $(1 \leqq i \leqq m)$. Σ is said to be *reduced* if each sentence in Σ is reduced with respect to Σ.

The following lemma can be proved in the same way as in the proof of Lemma 1 in [1; § 46].

Lemma 2. *Let Σ be a non-empty set of disjunctive universal sentences. Then, there exists a non-empty reduced set Γ of disjunctive universal sentences such that $\Sigma \Leftrightarrow \Gamma$ and any basic formula occurring in sentences in Γ occurs in some sentence in Σ.*

The next lemma is analogous to a part of Lemma 2 in [1; § 46].

Lemma 3. *Let Σ be a satisfiable reduced set of disjunctive universal sentences, and let*

$$\forall x_1 \cdots \forall x_n(\Theta_1 \vee \cdots \vee \Theta_m)$$

be a sentence in Σ, where $\Theta_1, \cdots, \Theta_m$ are basic formulas. Then for each i $(1 \leqq i \leqq m)$, there exist a structure $\mathfrak{A}$ in $\mathcal{M}(\Sigma)$ and elements $a_1, \cdots, a_n$ in $D[\mathfrak{A}]$ such that

$$(\dagger) \qquad \mathfrak{A} \models \Theta_i[a_1/x_1, \cdots, a_n/x_n]$$

and

$$(\dagger\dagger) \qquad \mathfrak{A} \models \neg\Theta_j[a_1/x_1, \cdots, a_n/x_n] \qquad \text{for all } j \neq i.$$

Proof. If $m=1$, it is obvious that (†) holds for any structure $\mathfrak{A}$ in the non-empty class $\mathcal{M}(\Sigma)$ and for any elements $a_1, \cdots, a_n$ in $D[\mathfrak{A}]$, and (††) is vacuous. If $m>1$, the assertion can be obtained in the same way as in the first part of the proof of Lemma 2 in [1; § 46].

The next theorem can be regarded as the converse of Theorem 1.

Theorem 2. *Let Σ be a satisfiable reduced set of disjunctive universal sentences. If $\mathcal{M}(\Sigma)$ is closed under the formation of zero-element extensions, then every sentence in Σ is regular.*

Proof. Assume that there exists some sentence Φ in Σ which is not regular. We shall prove that $\mathcal{M}(\Sigma)$ is not closed under the formation of zero-element extensions. Now let

$$\Phi = \forall x_1 \cdots \forall x_n(\Theta_1 \vee \cdots \vee \Theta_m),$$

where $\Theta_1, \cdots, \Theta_m$ are basic formulas. And let y be a variable not contained in $\{x_1, \cdots, x_n\}$. Then by the definition of regularity of disjunctive universal sentences, there exists a non-empty subset X of $\{x_1, \cdots, x_n\}$ which satisfies the following two conditions:

(1) y occurs in some $\Theta_{i_0}^{X \to y}$ $(1 \leqq i_0 \leqq m)$;

(2) For every j $(1 \leqq j \leqq m)$, $\Theta_j^{X \to y}$ is not regular with respect to y.

Now by Lemma 3, there exist a structure $\mathfrak{A}$ in $\mathcal{M}(\Sigma)$ and elements $a_1, \cdots, a_n$ in $D[\mathfrak{A}]$ such that

$$\mathfrak{A} \models \Theta_{i_0}[a_1/x_1, \cdots, a_n/x_n]$$

and

$$(\natural) \qquad \mathfrak{A} \models \neg\Theta_j[a_1/x_1, \cdots, a_n/x_n] \qquad \text{for all } j \neq i_0.$$

Let $\mathfrak{A}^*$ be a zero-element extension of $\mathfrak{A}$, and let $D[\mathfrak{A}^*] = D[\mathfrak{A}] \cup \{o\}$. Then by Lemma 1 (ii), it follows from (1) and (2) that

$$\mathfrak{A}^* \models \neg\Theta_{i_0}^{X \to y}[o/y, a_1/x_1, \cdots, a_n/x_n].$$

Moreover we have that for all $j \neq i_0$,

$$(\natural\natural) \qquad \mathfrak{A}^* \models \neg\Theta_j^{X \to y}[o/y, a_1/x_1, \cdots, a_n/x_n].$$

(For, if y does not occur in $\Theta_j^{X \to y}$ then ($\natural\natural$) follows from ($\natural$). If y occurs in $\Theta_j^{X \to y}$ then by Lemma 1 (ii), ($\natural\natural$) follows from (2).) Hence $\mathfrak{A}^* \models \Phi^{X \to y}$ does not hold. Therefore $\mathfrak{A}^* \models \Phi$ does not hold, because $\Phi \models \Phi^{X \to y}$. Hence $\mathcal{M}(\Sigma)$ is not closed under the formation of zero-element extensions. This completes the proof.

Theorem 3. *Let Φ be a satisfiable universal sentence. Then, Φ is preserved under the formation of zero-element extensions if and only if it is equivalent to a conjunction of regular disjunctive universal sentences.*

Proof. Since the "if" part follows immediately from Theorem 1, we shall prove the "only if" part. Obviously Φ is equivalent to a non-empty finite set of disjunctive universal sentences. Therefore by Lemma 2, there exists a non-empty finite reduced set Σ of disjunctive universal sentences such that $\Phi \Leftrightarrow \Sigma$. Hence Φ is equivalent to the conjunction of all sentences in Σ. Since $\mathcal{M}(\Phi) = \mathcal{M}(\Sigma)$ and it is closed under the formation of zero-element extensions, it follows by Theorem 2 that every sentence in Σ is regular. This completes the proof.

Theorem 4. *A positive universal sentence is preserved under the formation of zero-element extensions if and only if it is equivalent to a conjunction of weakly regular positive disjunctive universal sentences.*

Proof. It is obvious that a positive disjunctive universal sentence is regular if and only if it is weakly regular. And obviously, every positive universal sentence is satisfiable and is equivalent to a non-empty finite set of positive disjunctive universal sentences. Hence the theorem can be obtained in the same way as in the proof of Theorem 3.

Reference

[1] G. Grätzer: Universal Algebra. D. Van Nostrand Company, Inc. Princeton (1968).