88. The Baire Category Theorem in Ranked Spaces

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In this note, we study the Baire category theorem for a ranked space of indicator ω_0 (ω_0 is the first nonfinite ordinal). Throughout this note, the term "ranked space" will mean a ranked space of indicator ω_0. Terminologies and notations concerning ranked spaces will be the same as in [5], in particular, N will denote the set $\{0, 1, 2, \cdots\}$, $V(p)$, $W(p), \cdots$ preneighborhoods of p, and $V(p, n)$, $W(p, n), \cdots$ those of rank n of p.

1. The Baire category theorem. For a ranked space, we define the notion of nowhere dense as follows.

Definition 1. Let $(E, \mathcal{V})$ be a ranked space. A subset A of E is said to be *nowhere dense* in E if, for every $V(p) \in \mathcal{V}$, there exists a $V(q) \in \mathcal{V}$ such that $V(q) \subset V(p)$ and $V(q) \cap A = \phi$.

Moreover, as in [2] we define:

Definition 2. For a ranked space $(E, \mathcal{V})$, a subset A of E is said to be of *first category* if it is a countable union of nowhere dense sets. All other subsets of E are said to be of *second category*. A subset A of E is said to be *dense* in E if, for every $V(p) \in \mathcal{V}$, we have $V(p) \cap A \neq \phi$. The ranked space $(E, \mathcal{V})$ is called a *Baire space* if, for every subset A of E which is of first category, the complement $E - A$ is dense in E.

As is easily seen, if $(E, \mathcal{V})$ is a ranked space for which we can topologise E in such a way that the family of all sets belonging to $\mathcal{V}$ is a base of neighborhoods, then the notion of Baire category in $(E, \mathcal{V})$ coincides with that in the topological space E topologised in this way.

We first prove the following theorem.

Theorem 1. *Every complete ranked space is a Baire space.*

Already, for a ranked space whose indicator is an arbitrary inaccessible limit ordinal, the same theorem has been proved by K. Kunugi [2], [4] under the assumption that the family $\mathcal{V}$ of preneighborhoods in the ranked space satisfies the following conditions (B) and (C).

(B) For every $V_1(p), V_2(p) \in \mathcal{V}$, there exists a $V_3(p) \in \mathcal{V}$ such that $V_3(p) \subset V_1(p) \cap V_2(p)$.

(C) For every $V(p) \in \mathcal{V}$, if $q \in V(p)$, then there exists a $V(q) \in \mathcal{V}$ such that $V(q) \subset V(p)$.

Theorem 1 asserts that if we define nowhere dense as in Definition

1 and if the indicator of the ranked space is ω_0, then Kunugi's result holds without the assumptions of (B) and (C).

Proof of Theorem 1. Let $(E, \mathcal{V})$ be a complete ranked space. Let $A=\bigcup_{i=1}^{\infty} H_i$, where each H_i is nowhere dense in E, and let $V(p) \in \mathcal{V}$. We will show that $V(p) \cap (E-A) \neq \phi$. We first put $G_i = E - H_i$ for all i. Then, since H_0 is nowhere dense in E, there exists a $V(q_0) \in \mathcal{V}$ such that $V(q_0) \subset V(p)$ and $V(q_0) \subset G_0$. Also, by the axiom (a) of ranked space, there exists a $V(q_0, n_0) \in \mathcal{V}$ such that $V(q_0, n_0) \subset V(q_0)$. Thus, for $V(p)$, we may take a $V(q_0, n_0)$ such that $V(q_0, n_0) \subset V(p) \cap G_0$. Moreover, by the axiom (a), we may take a $V(q_1, n_1) \in \mathcal{V}$ such that $V(q_1, n_1) \subset V(q_0, n_0)$, $q_1 = q_0$ and $n_1 > n_0$. Suppose that $V(q_j, n_j)$ ($j=0, 1, 2, \cdots, 2i-1$) have been chosen such that $V(q_0, n_0) \supset V(q_1, n_1) \supset \cdots \supset V(q_{2i-1}, n_{2i-1})$, $q_{2j} = q_{2j+1}$ for $0 \leq j \leq i-1$, $n_0 < n_1 < \cdots < n_{2i-1}$, and $V(q_{2j}, n_{2j}) \subset V(p) \cap G_j$ for $0 \leq j \leq i-1$. Then, since H_i is nowhere dense in E, we may take, as in the case of $i=0$, a $V(q_{2i}, n_{2i}) \in \mathcal{V}$ such that $V(q_{2i}, n_{2i}) \subset V(q_{2i-1}, n_{2i-1}) \cap G_i$ and $n_{2i} > n_{2i-1}$, and a $V(q_{2i+1}, n_{2i+1})$ such that $V(q_{2i+1}, n_{2i+1}) \subset V(q_{2i}, n_{2i})$, $q_{2i+1} = q_{2i}$ and $n_{2i+1} > n_{2i}$. We thus obtain a fundamental sequence $\{V(q_i, n_i)\}$ such that $\cap V(q_i, n_i) \subset V(p) \cap (\cap G_i)$. Hence, $V(p) \cap (E-A) \neq \phi$ follows from the completeness of $(E, \mathcal{V})$.

Example 1 (due to K. Kunugi [4]). Let R^2 be the 2-dimensional Euclidean space and let $p \in R^2$, $p=(x_0, y_0)$. For each $n \in N$ and for each real number l such that $2 \leq l < +\infty$, we denote by $V(p; n, l)$ the set $\{(x, y); 0 \leq (x-x_0)(y-y_0) < 1/n+1, 0 \leq x-x_0 < l, 0 \leq y-y_0 < l\}$, by $\mathcal{V}_n(p)$ the family of all $V(p; n, l)$ such that $2 \leq l < +\infty$, and by $\mathcal{V}(p)$ the family $\cup\{\mathcal{V}_n(p); n \in N\}$. Then, $(R^2, \mathcal{V}, \mathcal{V}_n)$, where $\mathcal{V} = \cup\{\mathcal{V}(p); p \in R^2\}$ and $\mathcal{V}_n = \cup\{\mathcal{V}_n(p); p \in R^2\}$, is a complete ranked space which does not satisfy (C*) (see **2** below) weaker than (C).

2. Characterizations of Baire spaces. We give some definitions which are needed for other characterizations of Baire spaces.

Definition 3. Let $(E, \mathcal{V})$ be a ranked space, and let A be a subset of E. Then, A is called *open* if, for every $p \in A$, there exists a $V(p) \in \mathcal{V}$ such that $V(p) \subset A$. A is called *closed* if $E-A$ is open. The set $\cup\{O; O \text{ is open}, O \subset A\}$ is called the *interior* of A and denoted by A^i. The set $\cap\{F; F \text{ is closed}, A \subset F \subset E\}$ is called the *closure* of A and denoted by A^a.

Moreover, for $(E, \mathcal{V})$, we consider the following condition.

(C*) For every $V(p) \in \mathcal{V}$, there exists a $W(p) \in \mathcal{V}$ such that $W(p) \subset V(p)$ and such that, for every $q \in W(p)$, there exists a $V(q) \in \mathcal{V}$ such that $V(q) \subset V(p)$.

Then, we have

Proposition 1. *If, for a ranked space $(E, \mathcal{V})$, $\mathcal{V}$ satisfies (C*), then a subset A of E is nowhere dense in E if and only if A^a is nowhere* *dense in* E.

Proposition 2. *For a ranked space* $(E, \mathcal{V})$, *let us consider the following.*

(α) $(E, \mathcal{V})$ *is a Baire space.*

(β) *Every countable intersection of open dense sets in* E *is dense in* E.

(γ) *For every countable family* F_n $(n=1, 2, \cdots)$ *of closed sets satisfying* $E = \bigcup F_n$, $\bigcup (F_n)^i$ *is dense in* E.

Then, we have: (1) *If* $\mathcal{V}$ *satisfies* (B) *and* (C*), *then* (α) *implies* (β) *and* (γ); (2) *If* $\mathcal{V}$ *satisfies* (C*), *then each of* (β) *and* (γ) *implies* (α).

The proofs of these propositions are similar to those of the corresponding results in topological spaces.

3. Complete ranked spaces and α-favorable topological spaces (due to G. Choquet [1]). As a technique for deciding when a given topological space is Baire, G. Choquet [1] has introduced the notion of α-favorable, stemming from game theory, and proved that every α-favorable topological space is a Baire space. The following proposition shows the connection between the notion of completeness in ranked spaces and the notion of α-favorable.

Proposition 3. *Let* E *be a topological space for which we can define a complete ranked space* $(E, \mathcal{V})$ *such that* (1): $\mathcal{V}$ *is a family consisting of neighborhoods in* E *which forms a base for the topology of* E, *furthermore* $\mathcal{V}$ *has the property* (2): *there exists a* $k \in N$ *such that if, for* $V(p,n), V(q,m) \in \mathcal{V}$, $V(p,n) \supset V(q,m)$ *and* $V(q,m) \neq \{q\}$, *then* $n \leq m+k$. *Then,* E *is* α*-favorable.*

Proof. We define a map f of $\mathcal{V}$ into $\mathcal{V}^{*}$[*] in such a way that: if $V(p) \in \mathcal{V}$, then $f(V(p))$ is a $V(p,n) \in \mathcal{V}$ for which there exists a $V(p,m) \in \mathcal{V}$ such that $V(p,n) \subset V(p,m) \subset V(p)$ and $m+k<n$. The existence of such a $V(p,n)$ follows from the axiom (a) of ranked space. We will prove that if $\{V(p_{2i}); i=0,1,2,\cdots\}$ is a sequence of neighborhoods defined inductively so that

$$V(p_0) \supset V(p_1) = f(V(p_0)) \supset V(p_2) \supset V(p_3) = f(V(p_2)) \supset \cdots,$$

then $\bigcap V(p_{2i}) \neq \phi$. We put $f(V(p_{2i})) = V(p_{2i}, n_{2i})$. Then, we may obtain a sequence $\{V(p_{2i}, m_{2i}); i=0,1,2,\cdots\}$ of neighborhoods such that (1°): $m_{2i}+k<n_{2i}$ for all i, and such that $V(p_{2i}, n_{2i}) \subset V(p_{2i}, m_{2i}) \subset V(p_{2i})$ for all i, and therefore (2°): $V(p_0, m_0) \supset V(p_0, n_0) \supset \cdots \supset V(p_{2i}, m_{2i}) \supset V(p_{2i}, n_{2i}) \supset \cdots$. In (2°), if $V(p_{2i}, m_{2i}) \neq \{p_{2i}\}$ for all i, then by (2) and (1°), we have $m_0+k<n_0 \leq m_2+k<n_2 \leq \cdots$, and so a subsequence of (2°) is fundamental. Hence, $\bigcap V(p_{2i}, n_{2i}) \neq \phi$. If, in (2°), there exists an i_0 such that $V(p_{2i_0}, m_{2i_0}) = \{p_{2i_0}\}$, then $\bigcap V(p_{2i}, n_{2i}) = \{p_{2i_0}\}$. Thus, $\bigcap V(p_{2i}) \neq \phi$ follows.

*) We remark that [1], 7.13 holds under the assumption that $\mathcal{T}^*$ in [1], 7.11 is a base of neighborhoods.

The following examples are topological spaces satisfying the assumptions of Proposition 3.

Example 2. Complete metric spaces.

Let E be a complete metric space with a distance function d and let $p \in E$. We denote the set $\{q \in E; d(p,q) < 1/2^n\}$ by $S(p,n)$. If p is an isolated point of E, we put $V(p)=\{p\}$ and define $\mathcal{V}_n(p)=\{V(p)\}$ for all $n \in N$. If not, there exists a subsequence of $N: n_0(p) < n_1(p) < \cdots < n_k(p) < \cdots$ such that $n_0(p)=0$ and such that, for every k, $S(p, n_{k+1}(p))$ is a proper subset of $S(p, n_k(p))$ and $S(p, n_k(p)) = S(p,n)$ for all $n_k(p) \leq n < n_{k+1}(p)$. Using $\{n_k(p)\}$, we define $\mathcal{V}_n(p)$ as follows. For $n \in N$, if $n = n_k(p)$ for some $k \in N$, $\mathcal{V}_n(p) = \{S(p,n)\}$, that is, $S(p,n)$ is the only preneighborhood of rank n of p; otherwise, $\mathcal{V}_n(p) = \phi$. Then, $(E, \mathcal{V}, \mathcal{V}_n)$ is a desired ranked space if we put $\mathcal{V} = \cup \{\mathcal{V}(p); p \in E\}$, where $\mathcal{V}(p) = \cup \{\mathcal{V}_n(p); n \in N\}$, and put $\mathcal{V}_n = \cup\{\mathcal{V}_n(p); p \in E\}$ (cf. [3], Theorem 1).

Example 3. Cartesian products of the real lines, endowed with the product topology.

In this case, the ranked space obtained by putting $V(x_1, x_2, \cdots, x_n; m) = \{f(x); |f(x_i)| < 1/2^m\}$ in [3], Example, is a desired ranked space.

References

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