172. *On Holomorphically induced Representations of Split Solvable Lie Groups*

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We shall give an answer to three open problems for holomorphically induced representations of split solvable Lie groups.

1. Let G be a simply connected split solvable Lie group with Lie algebra $\mathfrak{g}$, f a linear form on $\mathfrak{g}$, $\mathfrak{h}$ a positive polarization of $\mathfrak{g}$ at f, $\rho(f, \mathfrak{h})$ the holomorphically induced representation of G constructed from $\mathfrak{h}$ and let $\mathcal{H}(f, \mathfrak{h})$ be the space of $\rho(f, \mathfrak{h})$ [1]. In this note, we find a necessary and sufficient condition on $(f, \mathfrak{h})$ for the non-vanishing of $\mathcal{H}(f, \mathfrak{h})$. We then show that $\rho(f, \mathfrak{h})(\neq 0)$ is irreducible if and only if the Pukanszky condition is satisfied, and in this case $\rho(f, \mathfrak{h})$ is independent of $\mathfrak{h}$. For reducible $\rho(f, \mathfrak{h})$, we describe its decomposition into irreducible components.

The details will appear elsewhere.

2. For a real vector space V, we denote its dual by V^*. Let $\mathfrak{d}=\mathfrak{h}\cap\mathfrak{g}$, $\mathfrak{e}=(\mathfrak{h}+\bar{\mathfrak{h}})\cap\mathfrak{g}$ and let $\mathfrak{b}=\mathfrak{d}\cap\ker f$. $\mathfrak{d}$ and $\mathfrak{b}$ are ideals of $\mathfrak{e}$. Let $\tilde{\mathfrak{e}}=\mathfrak{e}/\mathfrak{b}$, $\mathfrak{z}=\mathfrak{d}/\mathfrak{b}$, $\pi: \mathfrak{e}\to\tilde{\mathfrak{e}}$ the natural projection, $f_0=f|\mathfrak{e}\in\mathfrak{e}^*$, $\tilde{\mathfrak{h}}=\pi(\mathfrak{h})$ and let $\tilde{f}\in(\tilde{\mathfrak{e}})^*$ such that $\tilde{f}\circ\pi=f_0$. We denote by $P^+(f, \mathfrak{g})$ the set of positive polarizations of $\mathfrak{g}$ at f. Then, as a corollary of the fundamental theorem for normal Kähler algebras [3], we have the following theorem.

Theorem 1. *$\tilde{\mathfrak{e}}$ can be decomposed into a semi-direct sum*

$$\tilde{\mathfrak{e}}=\mathfrak{n}+\mathfrak{m},\ \mathfrak{m}: \textit{subalgebra},\ \mathfrak{n}: \textit{ideal},$$

and this decomposition satisfies the following conditions:

Let $\mathfrak{h}_1=\tilde{\mathfrak{h}}\cap\mathfrak{n}^C$, $\mathfrak{h}_2=\tilde{\mathfrak{h}}\cap\mathfrak{m}^C$, $\tilde{f}_1=\tilde{f}|\mathfrak{n}\in\mathfrak{n}^*$ and let $\tilde{f}_2=\tilde{f}|\mathfrak{m}\in\mathfrak{m}^*$.

a) *$\mathfrak{n}$ is a Heisenberg algebra with center $\mathfrak{z}$ and $\mathfrak{h}_1\in P^+(\tilde{f}_1, \mathfrak{n})$.*

b) *$\mathfrak{h}_2\in P^+(\tilde{f}_2, \mathfrak{m})$ and $\mathfrak{h}_2+\bar{\mathfrak{h}}_2=\mathfrak{m}^C$, $\mathfrak{h}_2\cap\mathfrak{m}=\{0\}$. We define the linear operator j on $\mathfrak{m}$ by $j(X)=-iX$ if $X\in\mathfrak{h}_2$, $j(X)=iX$ if $X\in\bar{\mathfrak{h}}_2$. Then $(\mathfrak{m}, j)$ is a narmal j-algebra.*

Note that $\mathfrak{n}$ or $\mathfrak{m}$ may be $\{0\}$.

3. We put $S(X, Y)=\tilde{f}_2([X, jY])$ for $X, Y\in\mathfrak{m}$.

Theorem 2 (Pjateckiĭ-Šapiro [4]). *Let $\mathfrak{a}$ be the orthogonal complement of $\eta=[\mathfrak{m}, \mathfrak{m}]$ with respect to the form S. $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{m}$, $\mathfrak{m}=\mathfrak{a}+\eta$, and the adjoint representation of $\mathfrak{a}$ on η is real diagonalizable. Thus, we have a decomposition of η into root spaces: $\eta=\sum\eta^\alpha$, where $\alpha\in\mathfrak{a}^*$ and $\eta^\alpha=\{X\in\eta;\ [A, X]=\alpha(A)X$ for all $A\in\mathfrak{a}\}$. Let $\{\eta^{\alpha_i}\}$, $1\leq i\leq r$ be those root spaces η^α for which $j(\eta^\alpha)\subset\mathfrak{a}$.*

Then $\dim \eta^{\alpha_i}=1$ *and* $r=\dim \mathfrak{a}$ (*r is called the rank of* $\mathfrak{m}$). *If we order* $\alpha_1, \cdots, \alpha_r$ *in an appropriate way, then all the other roots are of the form*

$$\begin{array}{ll} 1/2(\alpha_m+\alpha_k),\quad 1/2(\alpha_m-\alpha_k), & 1\leq k<m\leq r, \\ 1/2\alpha_k, & 1\leq k\leq r \end{array}$$

(*not all possibilities need occur*). *Let* $H_0=\mathfrak{a}+\sum_{m>k}\eta^{1/2(\alpha_m-\alpha_k)}$, $H_{1/2}=\sum \eta^{1/2\alpha_k}$, $H_1=\sum_{m\geq k}\eta^{1/2(\alpha_m+\alpha_k)}$ *and let* U_i *be the nonzero element of* η^{α_i} *such that* $[jU_i, U_i]=U_i$. *We put* $s=\sum_{i=1}^{r} U_i$. *Then* $\alpha_k(jU_i)=\delta_{k,i}$, $\mathrm{Ad}\, js\,|\,H_0=0$, $\mathrm{Ad}\, js\,|\,H_{1/2}=1/2\,\mathrm{Id}$, $\mathrm{Ad}\, js\,|\,H_1=\mathrm{Id}$, $j(\eta^{1/2(\alpha_m-\alpha_k)})=\eta^{1/2(\alpha_m+\alpha_k)}$ *for* $m>k$, $j(\eta^{1/2\alpha_k})=\eta^{1/2\alpha_k}$ *and* $jX=[s,X]$ *for* $X\in H_0$.

4. We keep the notations of Theorems 1 and 2. We put $L_i=\sum_{j>i}\eta^{1/2(\alpha_j-\alpha_i)}$, $L_i'=\sum_{i>j}\eta^{1/2(\alpha_i-\alpha_j)}$, $p_i=\dim L_i'$, $q_i=\dim L_i$, $r_i=\dim \eta^{1/2\alpha_i}$, $f_i=\tilde{f}_2(U_i)$ and $W=\ker \tilde{f}_1\subset \mathfrak{n}$. Then, W is stable by $\mathrm{ad}_{\mathfrak{n}}\,\mathfrak{m}$, $\mathrm{ad}_W\,\mathfrak{a}$ is real diagonalizable and W can be decomposed into root spaces W^β with roots of the form $\beta=\pm\frac{\alpha_i}{2}$, 0 (not all possibilities need occur). We put $t_i=\dim W^{\alpha_i/2}$.

The non-vanishing of $\mathcal{H}(f,\mathfrak{h})$ is related to the existence of nonzero holomorphic functions on some Siegel domain which belong to the L_2-space with respect to some Radon measure, and the following theorem is based on the results of H. Rossi and M. Vergne on normal j-algebras [5].

Theorem 3. $\mathcal{H}(f,\mathfrak{h})\neq\{0\}$ *if and only if*

$$-2f_i-(p_i+1+1/2(q_i+r_i+t_i))>0 \qquad 1\leq i\leq r.$$

5. G acts on $\mathfrak{g}^*$ by the coadjoint representation and thus we have the orbit space $\mathfrak{g}^*/G$. We denote by $0(f)$ the orbit through f. For each orbit $\omega\in\mathfrak{g}^*/G$, we denote by $\hat{\rho}(\omega)$ the equivalence class of irreducible unitary representations of G associated to ω in the sense of Kirillov-Bernat. We put $D=\exp\mathfrak{d}$ and, for each subspace $\mathfrak{a}$ of $\mathfrak{g}$, put $\mathfrak{a}^\perp=\{g\in\mathfrak{g}^*;\, g\,|\,\mathfrak{a}=0\}$. We say that $\mathfrak{h}$ satisfies the Pukanszky condition if $D.f=f+\mathfrak{e}^\perp$.

Theorem 4. *Suppose* $\mathcal{H}(f,\mathfrak{h})\neq\{0\}$. *Then* $\rho(f,\mathfrak{h})$ *is irreducible if and only if* $\mathfrak{h}$ *satisfies the Pukanszky condition. In this case* $\rho(f,\mathfrak{h})\in\hat{\rho}(0(f))$. *In particular,* $\rho(f,\mathfrak{h})$ *is independent of* $\mathfrak{h}$.

The essential part of the proof consists in showing that if $\mathfrak{h}$ satisfies the Pukanszky condition, $\rho(f,\mathfrak{h})\in\hat{\rho}(0(f))$. To prove this we proceed by induction on $\dim\mathfrak{g}$ and the proof of the theorem in [2] remains valid except for the case: $\mathfrak{e}=\mathfrak{g}$ and there is no ideal $\mathfrak{a}\neq\{0\}$ in $\mathfrak{g}$ such that $f(\mathfrak{a})=0$. In this case we choose some $f'\in 0(f)\subset\mathfrak{g}^*$ and some $\mathfrak{h}'\in P^+(f',\mathfrak{g})$ which also satisfies the Pukanszky condition and $\mathfrak{e}'=(\mathfrak{h}'+\bar{\mathfrak{h}}')\cap\mathfrak{g}\subsetneqq\mathfrak{g}$. We then construct the intertwining operator between $\rho(f, \mathfrak{h})$ and $\rho(f', \mathfrak{h}')$.

6. We denote by $U(f, \mathfrak{h})$ the set of orbits $\omega \in \mathfrak{g}^*/G$ such that $\omega \cap (f + \mathfrak{e}^{\perp})$ is non-empty open set in $f+\mathfrak{e}^{\perp}$. For $\omega \in \mathfrak{g}^*/G$, we denote by $c(\omega, f, \mathfrak{h})$ the number of connected components of $\omega \cap (f+\mathfrak{e}^{\perp})$. Then we have the following theorem which was proved by M. Vergne [6] for real polarizations of exponential groups.

Theorem 5. *If* $\mathcal{H}(f, \mathfrak{h}) \neq \{0\}$, *then*

a) $U(f, \mathfrak{h})$ *is a finite set.*

b) *For* $\omega \in U(f, \mathfrak{h})$, $c(\omega, f, \mathfrak{h}) < +\infty$.

c) $\rho(f, \mathfrak{h}) \in \sum_{\omega \in U(f, \mathfrak{h})} c(\omega, f, \mathfrak{h}) \hat{\rho}(\omega)$.

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