16. An Interpolation of Operators in the Martingale H_p-spaces

By Masami OKADA
Mathematical Institute, Tôhoku University

(Comm. by Kôsaku YOSIDA, M. J. A., Feb. 12, 1976)

1. Introduction. In this note we show that the Marcinkiewicz interpolation theorem of operators can be extended in the martingale setting.

2. Definition. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_n\}_{n=1}^{\infty})$ a probability space furnished with a non-decreasing sequence of σ-algebras of measurable subsets $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \cdots \subset \mathcal{F} = \bigvee_{n=1}^{\infty} \mathcal{F}_n$.

We define the set of random variables $H_p = H_p(\Omega, \mathcal{F}, P, \{\mathcal{F}_n\}_{n=1}^{\infty}) = \left\{ f \in L^p(\Omega)\,;\, |||f|||_p = \left[\int_{\Omega} (f^*)^p dP\right]^{1/p} < \infty \right\}$, where $f^*(w) = \sup_{1 \leqq n < \infty} |f_n(w)|$ and $p \geqq 1$.

Note that $H_1 \subsetneqq L^1$, and that $H_p = L^p$ for $1 < p < \infty$. In fact, there exists a constant A_p such that $\|f\|_p \leqq |||f|||_p \leqq A_p \|f\|_p$. Furthermore, as is well-known, the norm $|||f|||_p$ is equivalent to $\|(\sum_{n=1}^{\infty} |\Delta f_n|^2)^{1/2}\|_p$, where $\Delta f_n = f_n - f_{n-1}$, $f_0 = 0$ ([1]–[3]).

3. Weak type result. Let T be an operator from H_p to the set of random variables defined on a σ-finite measure space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

Theorem. *Suppose that*

(1) *T is quasi-linear, i.e.* $|T(f+g)| \leqq C(|Tf| + |Tg|)$

(2) $\tilde{P}(\{w\,; |Tf(w)| > t\})^{1/q_i} \leqq M_i/t\, |||f|||_{p_i}$, *for all* $t > 0$, *where* $1 \leqq p_i \leqq q_i < \infty$ $(i=0,1)$, $p_0 \neq p_1$ *and* $q_0 \neq q_1$. *Let us put* $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$, $0 < \theta < 1$. *Then*

$$\|Tf\|_q \leqq AC(C+1) M_0^{1-\theta} M_1^{\theta} \|f\|_p,$$

where

$$A^q = 0((q_1 - q)^{-1} + (q - q_0)^{-1}(p-1)^{-q_0}).$$

Proof. We consider the case $1 = p_0 < p_1$ and $q_0 < q_1$ only, the other cases are treated similarly.

1-st step. The following decomposition lemma is used in the proof, which corresponds to the Calderón-Zygmund decomposition ([4]–[6]).

Lemma (R. Gundy). *Let $v \in L^1(\Omega)$, $r \geqq 1$. Then for each $a > 0$, v is decomposed into three random variables g, h, k, $v = g + h + k$, which satisfy*

(1) $P(\{w\,; g^*(w) > 0\}) \leqq K/a \|v\|_1, \qquad \|g\|_r \leqq K \|v\|_r$

(2) $\left\| \sum_{n=1}^{\infty} |\Delta h_n| \right\|_1 \leqq K \|v\|_1, \qquad \|h\|_1 \leqq K \|v\|_1$

(3) $\|k\|_\infty \leqq Ka, \qquad \|k\|_1 \leqq K\|v\|_1$

with a constant K independent of a, v, r.

Now put $\lambda = p_0(q-q_0)/q_0(p-p_0)$, $\rho = -q_0/(q_1-q_0)$, $\sigma = q_1/(q_1-q_0)$, $\tau = (p_1q - pq_1)/p_1(q-q_1)$, $B = M_0^\rho M_1^\sigma \|f\|_p^\tau$, $r = (p+1)/2$ (>1).

2-nd step. Let $f \in L^p(\Omega)$. Then for each $y > 0$ the following decomposition of f is possible;

(1) $f = u + u'$, $u' = k + g + h$

(2) $u = f$, if $|f| < (y/B)^\lambda$ and $u = 0$, elsewhere.

(3) There exists a constant K independent of y, u', r, so that

$$\|k\|_{p_1}^{p_1} \leqq K(y/B)^{\lambda(p_1-r)} \|u'\|_r^r$$

$$\||g\||_1 \leqq KA_r(y/B)^{\lambda(1-r)} \|u'\|_r^r$$

and

$$\||h\||_1 \leqq K(y/B)^{\lambda(1-r)} \|u'\|_r^r.$$

In fact (3) is shown by lemma as follows. Put $v = u'$ and $a = (y/B)^\lambda$ in the following inequalities.

$$\|k\|_{p_1}^{p_1} \leqq \int |k|\, dP \cdot \|k\|_\infty^{p_1-1} \leqq K\|v\|_1 a^{p_1-1} \leqq K \int |v|^r\, dP,$$

$$\||g\||_1 \leqq P(g^* > 0)^{1/r'} \left[\int (g^*)^r dP\right]^{1/r} \leqq (K/a\|v\|_1)^{1/r'} A_r \|g\|_r$$
$$\leqq K a^{r/r'} \cdot A_r \|v\|_r^{r/r'+1}$$

and

$$\||h\||_1 \leqq \left\| \sum_{n=1}^\infty |\Delta h_n| \right\|_1 \leqq K\|v\|_1 \leqq K a^{1-r} \|v\|_r^r.$$

3-rd srep. Considering the decomposition above, we may write

$$\|Tf\|_q^q = q \int_0^\infty y^{q-1} \tilde{P}(|Tf| > y) dy \leqq q(4c(c+1))^q (I_1 + I_2 + I_3 + I_4),$$

where

$$I_1 = \int_0^\infty y^{q-1} \tilde{P}(|Tu| > y) dy,$$

$$I_2 = \int_0^\infty y^{q-1} \tilde{P}(|Tk| > y) dy,$$

$$I_3 = \int_0^\infty y^{q-1} \tilde{P}(|Tg| > y) dy,$$

and

$$I_4 = \int_0^\infty y^{q-1} \tilde{P}(|Th| > y) dy.$$

Now the rest of the proof is almost the same as in [4]. For example, we estimate the value I_3 as follows.

$$I_3 \leqq M_0^{q_0} \int_0^\infty y^{q-q_0-1} \||g\||_1^{q_0}\, dy$$
$$\leqq K A_r^{q_0} M_0^{q_0} B^{\lambda(r-1)q_0} \int_0^\infty y^{q-q_0-1+\lambda(1-r)q_0} \|u'\|_r^{rq} dy$$
$$\leqq K A_r^{q_0} M_0^{q_0} B^{\lambda(r-1)q_0} \left[\int \left\{ \int_0^{B|f|^{1/\lambda}} y^{q-q_0-1+\lambda(1-r)q_0} |u'|^{q_0 r} dy \right\}^{1/q_0} dp\right]^{q_0}$$

$$\leqq KA_r^{q_0} M_0^{q_0} B^{q-q_0} / ((q-q_0) + \lambda(1-r)q_0) \left[\int |f|^{(q-q_0)/q_0\lambda+1} dP \right]^{q_0}$$

$$\leqq 0(1/(r-1)^{q_0}(q-q_0)) M_0^{(1-\theta)q} M_1^{\theta q} \|f\|_p^q \qquad \text{Q.E.D.}$$

4. Remarks.

(1) The result also holds even if $P(\Omega)=\infty$.

(2) If $X=(X_n)_{n=0}^{\infty}$ is a martingale with respect to $(\Omega, \mathcal{F}, P, \{\mathcal{F}_n\}_{n=1}^{\infty})$ we say that $X \in M_p$ $(1 \leqq p < \infty)$ when $\|X\|_{M_p} = \sup_{n \geqq 1} E(|X_n|^p)^{1/p} < \infty$. Then H_p is isomorphic to M_p for $1 \leqq p < \infty$ by the correspondence, $f(w) \leftrightarrow X_\infty(w) = \lim_{n\to\infty} X_n(w)$. Therefore it is concluded that the interpolation theorem of operators also holds on martingale spaces M_p for $1 \leqq p < \infty$.

References

[1] D. L. Burkholder and R. F. Gundy: Extrapolation and interpolation of quasi-linear operators on martingale transforms. Acta Math., **124**, 249–304 (1970).

[2] B. Davis: On the integrability of the martingable square function. Israel Journal Math., **8**, 187–190 (1970).

[3] A. Garsia: Martingale Inequalities. Benjamin, Inc. (1973).

[4] S. Igari: An extension of the interpolation theorem of Marcinkiewicz. Tôhoku Math. J., **15**, 343–358 (1963).

[5] E. M. Stein: Topics in Harmonic Analysis. Ann. Math. Studies, **63**. Princeton Univ. Press (1970).

[6] ——: Singular Integrals and Differentiability Properties of Functions. Princeton Univ. Press, Princeton, N. J. (1970).

[7] A. Zygmund: Trigonometric Series (2nd ed.). Cambridge Univ. Press, London (1968).