61. *On Higher Coassociativity*

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In this note, we generalize the coassociativity of co-H-spaces and primitive maps among them, and give a relation between A'_4-spaces and their coretractions. We work in the category of based spaces having the homotopy types of CW-complexes and based maps. Details will appear in [2].

Let K_n ($n \geqq 2$) be Stasheff's convex polyhedron [3], which admits face maps $\partial_k(r, s): K_r \times K_s \to K_n$, $r+s=n+1$, $1 \leqq k \leqq r$, and degeneracy maps $s_j: K_n \to K_{n-1}$, $1 \leqq j \leqq n$, satisfying suitable FD-commutativities. For a given based space X, $W_n(X)$ denotes the wedge product of n-copies of X, (i, x) denotes the element whose i-th factor is x.

Definition 1. A space X is an A'_n*-space*, if there exists an A'_n-*structure* $\{M'_{X,i}: X \times K_i \to W_i(X)\}_{2 \leqq i \leqq n}$ satisfying the following conditions:

(1.1) $\mu'_X = M'_{X,2}$ is a comultiplication with the counit $*: X \to *$, where $*$ is the base point of X;

(1.2) for any $(\rho, \sigma) \in K_r \times K_s$, $r+s=i+1$, it holds

$$M'_i(\ ;\partial_k(r, s)(\rho, \sigma)) = M'_s(\ ;\sigma)(k) \cdot M'_r(\ ;\rho),$$

where $M'_s(\ ;\sigma)(k)$ implies $M'_s(\ ;\sigma)$ is applied on the k-th factor and 1 is applied on other factors;

(1.3) for $i \geqq 3$, there exist homotopies

$$D'_{X,i,j}: M'_{i-1}(\ ;s_j(\tau)) \simeq p_j M'_i(\ ;\tau)$$

where $p_j = \nabla(j) \cdot *(j)$ and $\nabla: X \vee X \to X$ is the folding map.

Definition 2. An A'_n-space X ($n \geqq 3$) is an A'_n*-cogroup*, if there exists a coinversion $\nu'_X: X \to X$ such that it holds $\nabla \cdot (1 \vee \nu'_X) \cdot \mu'_X \simeq * \simeq \nabla \cdot (\nu'_X \vee 1) \cdot \mu'_X$.

Definition 3. A map $f: X \to Y$ of A'_n-spaces is an A'_n*-map* if there exist homotopies $H'_i: X \times K_i \times \boldsymbol{I} \to W_i(Y)$, $2 \leqq i \leqq n$, such that

(3.1) $H'_i((x;\tau), 0) = W_i(f) \cdot M'_{X,i}(x;\tau)$

and

$$H'_i((x;\tau);1) = M'_{Y,i}(f(x);\tau);$$

(3.2) for any $\partial_k(r, s)$, $r+s=i+1$, $1 \leqq k \leqq r$,

there exists a homeomorphism $\tilde{\partial}_k(r, s)$ of $K_r \times K_s \times \boldsymbol{I}$ into $\partial K_i \times \boldsymbol{I}$ which preserves level and satisfies

$$H'_i(\tilde{\partial}_k(r,s)(x;(\rho,\sigma),t)$$
$$=\begin{cases} H'_s\left((\ ;\sigma),\dfrac{(2^{t-1}-1)t}{2^{s-1}-1}\right)(k)\cdot M'_{X,r}(x;\rho) & \text{for } 0\leqq t\leqq \dfrac{2^{s-1}-1}{2^{t-1}-1} \\ M'_{Y,s}(\ ;\sigma)(k)\cdot H'_r\left((x;\rho),\dfrac{(2^{t-1}-1)t+1-2^{s-1})}{2^{t-1}-2^{s-1}}\right) & \text{for } \dfrac{2^{s-1}-1}{2^{t-1}-1}\leqq t\leqq 1; \end{cases}$$

(3.3) there exists a homotopy $G: X\times I\times I\to Y\times Y$ such that $G(x,t,0)=(f\times f)\cdot D'_{X,t}(x,t)$, $G(x,1,s)=j\cdot H'_2(x,s)$, $G(x,t,1)=D'_Y(f(x),t)$ and $G(x,0,s)=\Delta_Y\cdot f(x)$, where $j: X\vee X\to X\times X$ is the inclusion map, Δ is the diagonal map and D' is the homotopy from Δ to $j\cdot\mu'$. (In this situation, we say that H'_2 is compatible with D'.)

(3.4) $\{H'_i\}$ are compatible with $D'_{X,i,j}$ and $D'_{Y,i,j}$.

A suspended space SA admits a canonical A'_n-structure $\{M'_{0,i}\}_{2\leqq i\leqq n}$ induced by suspension structure for all $n\geqq 2$, and the suspension Sf is an A'_n-map.

Let $\varepsilon: S\Omega X\to X$ be the evaluation map, i.e., $\varepsilon\langle a,l\rangle=l(a)$, then $\gamma: X\to S\Omega X$ is called a (homotopy) coretraction of X if $\varepsilon\cdot\gamma\simeq 1$. As is well known, the set of homotopy classes of coretractions and the set of comultiplications of X are in 1 to 1 correspondence, and γ is an A'_2-map if and only if X is an A'_3-cogroup (cf. [1]).

Definition 4. An A'_4-cogroup X is an *s-A'_4-cogroup*, if it holds

$$W_4(\varepsilon)\cdot(1\vee\nu'_0\vee 1\vee\nu'_0)\cdot(1\vee 1\vee\mu'_0)\cdot M'_{0,3}(\gamma\times 1)$$
$$\simeq(1\vee\nu'_X\vee 1\vee\nu'_X)(1\vee 1\vee\mu'_X)\cdot M'_{X,3}\quad \text{rel. } X\times\partial K_3.$$

Now, we obtain the following

Theorem 1. *Let X be an A'_3-cogroup, then X is an s-A'_4-cogroup if and only if the corresponding γ is an q-A'_3-map.*

Proof. *Sufficiency.* K_3 is a line-segment whose vertices correspond to $(\mu'\vee 1)\cdot\mu'$ and $(1\vee\mu')\cdot\mu'$, respectively, and K_4 is a pentagon whose vertices are $P_0((\mu'\vee\mu')\cdot\mu')$, $P_1((\mu'\vee 1\vee 1)\cdot(\mu'\vee 1)\cdot\mu')$, $P_2((1\vee\mu'\vee 1)\cdot(\mu'\vee 1)\cdot\mu')$, $P_3((1\vee\mu'\vee 1)\cdot(1\vee\mu')\cdot\mu')$ and $P_4((1\vee 1\vee\mu')\cdot(1\vee\mu)\cdot\mu')$. Define $H': X\times K_4\times\{1\}\cup(\cup\tilde{\partial}_k(r,s))\to W_4(X)$ using $M'_{0,4}$, H'_2 and H'_3. H' may be extended to H'' over $X\times K_4\times\{1\}\cup\partial K_4\times I$ using $H'_2\vee H'_2$. Extend H'' to M'' over $X\times K_4\times I$, and put $M'_{X,4}=M''|X\times K_4\times\{0\}$, which gives the desired structure.

Necessity. Define $\Phi_k: W_{k-1}(S\Omega X)\to W_k(X)$ by

$$\Phi_k(i,\langle a,l\rangle)=\begin{cases}(i,l(2a)) & \text{for } 0\leqq a\leqq \dfrac{1}{2}\\ (k,l(2-2a)) & \text{for } \dfrac{1}{2}\leqq a\leqq 1\end{cases}$$

then Φ_k is a homotopy-monomorphism, and put

$$\Pi_X=(1\vee\nu'_X\vee 1\vee\nu'_X)\cdot(1\vee 1\vee\mu'_X)\cdot M'_{X,3}.$$

Then, we shall have the following:

(i) $\Phi_4(1\vee\nu'\vee\nu')=(1\vee 1\vee T)\cdot\rho\cdot(\Phi_2\vee\Phi_2\vee\Phi_2)$, where T is the switching map and $\rho=(1\vee 1\vee \nabla \vee 1)\cdot(1\vee T\vee 1\vee 1)\cdot(1\vee \nabla \vee 1\vee 1\vee 1)$;

(ii) $\rho(\bar{\mu}'_X\vee \bar{\mu}'_X\vee \bar{\mu}'_X)\cdot M'_{X,3}\simeq \Pi_X$ rel. $X\times\partial K_3$,

where $\bar{\mu}'_X=(1\vee \nu'_X)\cdot \mu'_X$;

(iii) $\Phi_4(1\vee \nu'_0\vee \nu'_0)\cdot W_3(\gamma)\cdot M'_{X,3}\simeq(1\vee 1\vee T)\cdot \Pi_X$ rel. $X\times\partial K_3$; and

(iv) $\Phi_4(1\vee \nu'_0\vee \nu'_0)\cdot M'_{0,3}\simeq(1\vee 1\vee T)\cdot W_4(\varepsilon)\cdot \Pi_0$ rel. $X\times\partial K_3$.

Since X is an s-A'_4-cogroup, we shall obtain

$$\Phi_4(1\vee \nu'_0\vee \nu'_0)\cdot M'_{0,3}\cdot(\gamma\times 1)$$
$$\simeq \Phi_4(1\vee \nu'_0\vee \nu'_0)\cdot W_3(\gamma)\cdot M'_{X,3} \quad \text{rel. } X\times\partial K_3,$$

therefore, since Φ_4 is a homotopy monomorphism, γ is an q-A'_3-map.

References

[1] Ganea, T.: Cogroups and Suspensions. Inventiones Math., **9**, 185–197 (1970).

[2] Saito, S.: On Higher Coassociativity (to appear).

[3] Stasheff, J.: Homotopy associativity of H-spaces. I. Trans. Amer. Math. Soc., **108**, 275–292 (1963).