# 2. *Projective Differential-Geometrical Properties of the One-Parameter Families of Point-Pairs in the One-Dimensional Space.*

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1. When we consider point-pairs in the one-dimensional space as space element, we can treat a one-parameter family of point-pairs in quite similar way as a curve in the two-dimensional space.

In the homogeneous point-coordinates $(x_1, x_2)$ a point-pair is represented by the equation

$$\sum_{i,k} a_{ik} x_i x_k = 0 \qquad (i, k = 1, 2),$$

therefore we take $a_{ik}$ as the homogeneous coordinates of a point-pair, and

$$\overset{*}{a}_{ik} = |a_{lm}|^{-\frac{1}{2}} a_{ik}$$

as its *normalized coordinates.*

2. Let $a_{ik}(t)$ be the normalized coordinates of a one-parameter family $F$ of point-pairs, and let

$$(m, n) = \begin{vmatrix} a_{11}^{(m)} & a_{12}^{(n)} \\ a_{21}^{(m)} & a_{22}^{(n)} \end{vmatrix} + \begin{vmatrix} a_{11}^{(n)} & a_{12}^{(m)} \\ a_{21}^{(n)} & a_{22}^{(m)} \end{vmatrix},$$

where

$$a_{ik}^{(m)} = \frac{d^m a_{ik}(t)}{dt^m}.$$

Then evidently the relations

$$(0, 0) = 2, \qquad (1, 0) = 0$$

hold good, while $(1, 1)$ does not identically vanish in general. So as the natural parameter we adopt

$$p = \frac{1}{i\sqrt{2}} \int \sqrt{(1, 1)}\, dt$$

instead of $t$ and denote $(2, 2)$ by $2I$. We call the quantity $p$ the *projective* *length* and $I(p)$ the *projective curvature* of the family $F$, which are both invariant under the projective transformation group.

3. We can prove the fundamental theorem:

*When $I$ be given as a once continuously differentiable function $f(p)$ of the projective length $p$, the family of point-pairs with the projective curvature $I$ and the projective length $p$ is uniquely determined, except for the projective transformations.*

We consider, therefore, $I=f(p)$ as the natural equation of the family.

4. We can easily see that the point-pair $a'_{ik}(p)$ belongs to the involution, determined by the two point-pairs $a_{ik}(p)$ and $a_{ik}(p+dp)$, and is harmonic with the point-pair $a_{ik}(p)$. For the family of such a point-pair the projective length and the projective curvature are respectively

$$p_1 = -i\int I^{\frac{1}{2}}\,dp \quad \text{and} \quad I_1 = \frac{II'^2(3-4I)}{2(1-I)} + 2I^4.$$

5. Further we can prove the following theorems.

Theorem 1. *Two continuous point-sets of the family $F$ having the constant projective curvature correspond projectively to each other and all point-pairs represented by $a'_{ik}(p)$ belong to an elliptic or a hyperbolic involution, according as the curvature is negative or positive, and conversely.*

Theorem 2. *In the family $F$ having projective curvature $I=0$, the point-pairs represented by $a''_{ik}(p)$ consist always of two coincident points, and conversely.*

Theorem 3. *All point-pairs of the family $F$ having the projective curvature $I=1$ belong to an involution, and conversely.*

Theorem 4. *Every point-pair of the family $F$, for which $(1,1)$ identically vanishes, contains always a fixed point, and conversely.*

6. I shall here add the geometrical meaning of the projective curvature. If we consider a family $F$ having the constant projective curvature, we can see from Theorem 1, that there exist two point-pairs $P_1$, $P_2$ consisting respectively of two coincident points in the family $F$. The anharmonic ratios of the point-pairs of the family $F$ with regard to two points $P_1$, $P_2$ are all equal, and when we denote this anharmonic ratio by $k$, the projective curvature is equal to

$$I = -\frac{1}{4k}(k-1)^2.$$

This is a geometrical meaning of the projective curvature, whose sign is opposite to that of $k$.

7. We find in general as the canonical expansions for the family $F$

$$a_{11} = 1+p+\frac{1}{2!}p^2+\frac{1}{3!}I_0p^3+\cdots\cdots,$$

$$a_{22} = 1-p+\frac{1}{2!}p^2-\frac{1}{3!}I_0p^3+\cdots\cdots,$$

$$a_{12} = \qquad +\frac{1}{2!}\sqrt{1-I_0}\,p^2-\frac{I_0'}{3!\,2\sqrt{1-I_0}}p^3+\cdots\cdots,$$

where $I_0$ is the value of the projective curvature for $p=0$.